

# SPECTRAL SYNTHESIS FOR COADJOINT ORBITS OF NILPOTENT LIE GROUPS

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ABSTRACT. We determine the space of primary ideals in the group algebra  $L^1(G)$  of a connected nilpotent Lie group by identifying for every  $\pi \in \widehat{G}$  the family  $\mathcal{I}^\pi$  of primary ideals with hull  $\{\pi\}$  with the family of invariant polynomials of a certain finite dimensional subspace  $\mathcal{P}_Q^\pi$  of the space of polynomials  $\mathcal{P}(G)$  on  $G$ .

## 1. INTRODUCTION

Let  $G$  be a connected and simply connected nilpotent Lie group and let  $\mathfrak{g}$  be its Lie algebra. Let  $\pi \in \widehat{G}$  be an irreducible unitary representation of  $G$ . Then  $\pi$  defines an irreducible unitary representation of the convolution algebra  $L^1(G)$ , the space of the measurable functions  $f: G \rightarrow \mathbb{C}$  which are integrable with respect to Haar measure.

Every equivalence class of representations  $\pi \in \widehat{G}$  defines the primitive ideal  $\ker(\pi)$  of  $L^1(G)$ , and the mapping  $\widehat{G} \rightarrow \text{Prim}(G), \pi \mapsto \ker(\pi)$  is a bijection, since  $G$  is type I and  $*$ -regular. Furthermore, the fact that  $G$  is connected and has polynomial growth implies that there exists for every  $\pi \in \widehat{G}$  a unique minimal ideal  $j(\pi)$  with hull  $\{\pi\}$  which is contained in every closed twosided ideal  $I$  with  $\text{hull}(I) = \{\pi\}$  (see [Lu80]). It well known that the Schwartz space  $\mathcal{S}(G)$  has a dense intersection with  $\ker(\pi)$  (see [Lu83a]). This implies then that  $\ker(\pi)^N$  is contained in  $j(\pi)$  for  $N \in \mathbb{N}$  large enough and is dense in it (see [Lu83b]).

On the other hand Kirillov's orbit picture of the spectrum  $\widehat{G}$  of  $G$  tells us that every irreducible unitary representation  $\pi$  of  $G$  is associated with a coadjoint orbit  $\mathcal{O}_\pi \subset \mathfrak{g}^*$ . Since the data  $\ker(\pi)$  and  $j(\pi)$  are determined by  $\pi$  and hence by the Kirillov orbit  $\mathcal{O}_\pi$ , one can ask if the geometric structure of the orbit  $\mathcal{O}_\pi$  gives us some information on the structure of the algebra  $\ker(\pi)/j(\pi)$ . For instance, one would like to determine the set  $\mathcal{I}^\pi$  of all ideals  $I$  in  $L^1(G)$  with hull  $\{\pi\}$ , the so called primary ideals of  $L^1(G)$ . In the case where the orbit  $\mathcal{O}_\pi$  is flat, then  $\ker(\pi)/j(\pi) = \{0\}$ , i.e.,  $\pi$  is a set of spectral synthesis. If the group  $G$  is step 3, then the structure of the algebra  $\ker(\pi)/j(\pi)$  has been determined in [Lu83a]: The set  $\mathcal{I}^\pi$  of all twosided closed ideals of  $L^1(G)$  contained in  $\ker(\pi)$  and containing  $j(\pi)$  is in bijection with the set of translation invariant subspaces of a certain finite

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dimensional translation invariant space  $\mathcal{P}^\pi$  of polynomials on  $\mathfrak{g}$ . This space  $\mathcal{P}^\pi$  is determined by a weight type condition coming from the orbit  $\mathcal{O}_\pi$ .

In this paper we discover for every nilpotent Lie group  $G$  and every  $\pi \in \widehat{G}$  a finite dimensional translation invariant subspace  $\mathcal{P}_0^\pi = \mathcal{P}_0$  of polynomials on  $G$ , such that the subspace  $J_{\pi,0} := \{f \in \mathcal{S}(G) \mid pf \in \ker(\pi) \text{ for all } p \in \mathcal{P}_0^\pi\}$  is dense in  $j(\pi)$  (Theorem 2.11). This theorem then implies that there exist for every maximal projection  $Q = Q_\xi$  (see Definition 3.3) a subspace  $\mathcal{P}_Q^\pi$  of  $\mathcal{P}_0^\pi$  whose invariant subspaces are in bijection with the set  $\mathcal{I}^\pi$  (see Theorem 3.13). In particular this shows that the Schwartz space  $\mathcal{S}(G)$  has a dense intersection with every primary ideal of  $L^1(G)$  (see Theorem 3.14). We then study the case where the space  $\mathcal{P}_Q^\pi$  is itself translation invariant (see Theorem 3.22). Finally we consider the ideals that are  $L^\infty(G/N)$  invariant, where  $N = \exp \mathfrak{n}$  with  $\mathfrak{n}$  the ideal generated by the radicals of the elements in the coadjoint orbit corresponding to  $\pi$ , and then add some examples.

The more difficult problem, which is completely open, is to determine explicitly for groups of step  $\geq 4$  these spaces  $\mathcal{P}_Q^\pi$  and their relations with the geometric structure of the Kirillov-orbit  $\mathcal{O}_\pi$ . For this one needs precise estimates of the growth of the functions  $pc_{\xi,\eta}^\pi$ , which are products of polynomial functions  $p$  with smooth coefficient functions  $c_{\xi,\eta}^\pi$ . This means that the determination of  $\mathcal{P}_Q^\pi$  forces us to understand oscillatory Fourier integrals of the form

$$c_{\xi,\eta}(g) = \int_{\mathbb{R}^d} \xi(Q_\pi(g, z)) \bar{\eta}(z) e^{iP_\pi(g, z)} dz, \quad g \in G,$$

where  $\xi, \eta \in \mathcal{S}(\mathbb{R}^d)$  and  $P_\pi, Q_\pi : G \times \mathbb{R}^d \mapsto \mathbb{R}^d$  are some special polynomial mappings coming from the realization of  $\pi$  as monomial representation.

**1.1. Notations.** Let as above  $\mathfrak{g}$  be a nilpotent Lie algebra. We can realize the group  $G = \exp \mathfrak{g}$  as the Lie algebra  $\mathfrak{g}$  itself by using the Campbell-Baker-Hausdorff multiplication

$$X \cdot Y := X \cdot_{\text{CBH}} Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \cdots, \quad X, Y \in \mathfrak{g}.$$

The Haar measure of this group  $(\mathfrak{g}, \cdot_{\text{CBH}})$  is then Lebesgue measure on the real vector space  $\mathfrak{g}$ . Let  $\mathfrak{g}^*$  be the dual of  $\mathfrak{g}$  and denote by

$$\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \mapsto \mathbb{R}$$

the natural duality.

We denote by  $L^1(G)$  the space of functions  $f : G \rightarrow \mathbb{C}$  which are integrable with respect to Haar measure. Then  $L^1(G)$  is an involutive Banach algebra for the convolution product  $*$  and the involution  $*$

$$\begin{aligned} f * g(x) &= \int_G f(y) g(y^{-1}x) dy \\ f^*(x) &= \overline{f(x^{-1})}, \quad x \in G. \end{aligned}$$

Let also for a function  $f : G \rightarrow \mathbb{C}$

$$\check{f}(x) := f(x^{-1}), \quad x \in G.$$

In this paper  $\mathcal{S}(\mathfrak{g})$  will denote the Fréchet space of Schwartz functions on the real vector space  $\mathfrak{g}$ , i.e.,  $\mathcal{S}(\mathfrak{g})$  is the space of rapidly decreasing smooth functions

on  $\mathfrak{g}$  and  $\mathcal{S}(G) := \{f: G \mapsto \mathbb{C} \mid f \circ \log \in \mathcal{S}(\mathfrak{g})\}$ . The subspace  $\mathcal{S}(G)$  is then a dense involutive subalgebra of  $L^1(G)$ .

The dual space  $\mathcal{S}'(G)$  of  $\mathcal{S}(G)$  is the space of temperate distributions on  $G$ . Then  $\mathcal{S}(G)$  is a subspace of  $BC^\infty(G)$ , the space of smooth bounded functions on  $G$ , with bounded left and right derivatives, and  $BC^\infty(G) \subset \mathcal{S}'(G)$ .

For a mapping  $\varphi$  from  $G$  into a set  $X$ , denote for  $g \in G$  by  $\lambda(g)\varphi$  the left translate and by  $\rho(g)\varphi$  the right translate of  $\varphi$ .

Let  $\mathcal{P}(G)$  be the space of all polynomial functions on  $G$  with complex coefficients, that is, functions  $p: G \rightarrow \mathbb{C}$  that are polynomials in any polynomial chart of  $G$ . For a  $p \in \mathcal{P}(G)$  we denote by  $\mathcal{V}_p$  the subspace of  $\mathcal{P}(G)$  generated by  $G$ -left and right translates of  $p$ ,

$$(1.1) \quad \mathcal{V}_p = \text{span}\{\lambda(x)\rho(y)p \mid x, y \in G\}.$$

This is a finite dimensional subspace of  $\mathcal{P}(G)$ , and a  $G \times G$  submodule of  $\mathcal{P}(G)$  for the action  $\lambda \otimes \rho$ . For  $p \in \mathcal{P}(G)$  the dimension of  $\mathcal{V}_p$  is called the degree or  $G$ -degree of  $p$ .

**1.2. Kirillov-theory.** The orbit picture of the spectrum  $\widehat{G}$  of  $G$ , discovered by Kirillov in [Ki62], describes the irreducible representations  $(\pi, \mathcal{H}_\pi)$  of the group  $G$  as induced representations. For every  $\ell \in \mathfrak{g}^*$ , there exists a polarization  $\mathfrak{p} \subset \mathfrak{g}$  at  $\ell$ , i.e.,  $\mathfrak{p}$  is a maximal isotropic subspace for the bilinear form  $B_\ell(X, Y) = \langle \ell, [X, Y] \rangle$ ,  $X, Y \in \mathfrak{g}$ , and at the same time a subalgebra of  $\mathfrak{g}$ . To  $\mathfrak{p}$  and  $\ell$  one associates the induced representation  $\pi_{\ell, \mathfrak{p}} = \text{ind}_P^G \chi_\ell$ , where  $P = \exp \mathfrak{p}$  is the closed connected subgroup of  $G$  with Lie algebra  $\mathfrak{p}$ , and  $\chi_\ell(\exp X) := e^{-i\langle \ell, X \rangle}$ ,  $X \in \mathfrak{p}$  is the unitary character of  $P$  whose differential is  $-i\ell|_{\mathfrak{p}}$ . For any polarization  $\mathfrak{p}$  at some  $\ell \in \mathfrak{g}^*$ , the representation  $\pi_{\ell, \mathfrak{p}}$  is irreducible. Finally, two irreducible representations  $\pi_{\ell, \mathfrak{p}}$  and  $\pi_{\ell', \mathfrak{p}'}$  are equivalent if and only if  $\ell$  and  $\ell'$  are contained in the same coadjoint orbit. Then for every irreducible representation  $(\pi, \mathcal{H}_\pi)$  of  $G$  there is an  $\ell \in \mathfrak{g}^*$  and a polarization  $\mathfrak{p}$  at  $\ell$  such that  $\pi$  is equivalent to  $\pi_{\ell, \mathfrak{p}}$ .

Thus there exists a bijection between the space of coadjoint orbits  $\mathfrak{g}^*/G$  and the spectrum  $\widehat{G}$  of  $G$  given by the Kirillov mapping

$$\mathcal{K}: \mathcal{O} \in \mathfrak{g}^*/G \mapsto \widehat{G}, \quad \mathcal{K}(\mathcal{O}) = [\pi_{\ell, \mathfrak{p}}],$$

where  $[\pi]$  denotes the unitary equivalence class of the representation  $\pi$ . This is actually an homeomorphism (see [Br73]).

### 1.3. Ideals in $L^1(G)$ .

For simplicity of notations, an ideal in this paper is always closed and twosided. If  $I \subset L^1(G)$  is a subspace, not necessarily closed, which is invariant under left and right multiplication by elements of  $L^1(G)$  we say that  $I$  is an algebra ideal.

Let  $\text{Prim}(G)$  be the space of the primitive ideals of the Banach algebra  $L^1(G)$  equipped with the Jacobson topology. Then the mapping

$$\widehat{G} \rightarrow \text{Prim}(G), \quad \pi \mapsto \ker(\pi) \subset L^1(G),$$

is a homeomorphism (see [BLSV78]). Every primitive ideal  $I \subset L^1(G)$  is maximal and every maximal ideal  $M \subset L^1(G)$  is primitive (see [Di60]).

**Definition 1.1.** i) For an ideal  $I$  of  $L^1(G)$  denote by  $\text{hull}(I)$  the (closed) subset of  $\widehat{G}$  defined by

$$\text{hull}(I) := \{\pi \in \widehat{G} \mid \pi(I) = \{0\}\}.$$

ii) For a closed subset  $C \subset \widehat{G}$  the kernel of  $C$  is the ideal

$$\ker(C) := \{f \in L^1(G) \mid \pi(f) = 0 \text{ for all } \pi \in C\}.$$

Let  $C \subset \widehat{G}$  be a closed subset. Then there exists a *minimal (algebra) ideal*  $J(C)$  in  $L^1(G)$  with hull  $C$ . This means that there exists a (unique) twosided algebra ideal  $J(C)$  in  $L^1(G)$ , which has the property that  $\pi(J(C)) = \{0\}$  if and only if  $\pi \in C$  and which is contained in every algebra ideal  $I$  of  $L^1(G)$  with  $\text{hull}(I) \subset C$ . This minimal ideal  $J(C)$  is generated by all the self-adjoint Schwartz functions  $f \in \mathcal{S}(G)$  for which there exists a Schwartz function  $g \in \ker(C)$  such that

$$g * f = f * g = f.$$

(see [Lu80]).

Let

$$j(C) := \overline{J(C)}^{\|\cdot\|_1}$$

be the closure in  $L^1(G)$ . Then  $j(C)$  is contained in every ideal  $I \subset L^1(G)$  with  $\text{hull}(I) \subset C$ .

Let now  $C = \{\pi\}$  be a singleton. It had been shown in [Lu83b] that for  $N \in \mathbb{N}$  large enough the ideal  $\ker(\pi)^N$  is contained in  $j(\pi)$  and therefore dense in  $j(\pi)$ . Let now  $N_\pi$  be the smallest such an integer. Then

$$(1.2) \quad \ker(\pi)^{N_\pi-1} \not\subset j(\pi), \quad \overline{\ker(\pi)^{N_\pi}}^{\|\cdot\|_1} = j(\pi).$$

#### 1.4. Multiplication of convolution products by polynomials.

Let  $p \in \mathcal{P}(G)$ . We choose a Jordan-Hölder basis  $\mathcal{X} = \{p = p_m, \dots, p_1\}$  of the  $G \times G$  submodule  $\mathcal{V}_p$  of  $\mathcal{P}(G)$  generated by  $p$ . This means that  $\mathcal{X}$  is a basis of  $\mathcal{V}_p$  and that

$$(1.3) \quad \begin{aligned} \lambda(s)p_j(t) &= \sum_{i=j}^1 a_{i,j}(s)p_i(t), \\ \rho(s)p_j(t) &= \sum_{i=j}^1 b_{i,j}(s)p_i(t), \end{aligned}$$

where the functions  $s \rightarrow a_{i,j}(s)$ ,  $b_{i,j}(s)$  are polynomial functions and  $a_{j,j}(s) = b_{j,j}(s) = 1$  for all  $s \in G$  and  $j = 1, \dots, m$ .

Let  $f \in \mathcal{S}(G)$  and  $g: G \rightarrow \mathbb{C}$  be a continuous polynomially growing function. Then for  $x \in G$  one has

$$\begin{aligned} p(x)(f * g)(x) &= p(x) \int_G f(s)g(s^{-1}x)ds \\ &= \int_G p(ss^{-1}x)f(s)g(s^{-1}x)ds \\ &= \sum_{i=1}^m \int_G f(s)p_i(s)b_{i,1}(s^{-1}x)g(s^{-1}x)ds \end{aligned}$$

and similarly

$$\begin{aligned}
p(x)(f * g)(x) &= p(x) \int_G f(s)g(s^{-1}x)ds \\
&= \int_G p(ss^{-1}x)f(s)g(s^{-1}x)ds \\
&= \sum_{i=1}^m \int_G f(s)a_{i,1}(s^{-1})p_i(s^{-1}t)g(s^{-1}x)ds.
\end{aligned}$$

Hence

$$\begin{aligned}
(1.4) \quad p(f * g) &= (pf) * g + \sum_{i=m-1}^1 (p_i f) * (b_{i,m}g) \\
&= f * (pg) + \sum_{i=m-1}^1 (\check{a}_{i,m}f) * (p_i g).
\end{aligned}$$

## 2. THE MINIMAL IDEAL

**2.1. A class of polynomials given by the growth of coefficients.** For a fixed  $\ell \in \mathfrak{g}^*$ , let  $\pi = \pi_\ell: G \mapsto \mathbb{B}(\mathcal{H}_\pi)$  be the (unique up to unitary equivalence) unitary irreducible representation corresponding to the coadjoint orbit  $\mathcal{O}_\ell = \text{Ad}^*(G)\ell$  of  $\ell$ . Then let  $\mathcal{H}_\pi^\infty$  be the space of smooth vectors for  $\pi$ , and  $\mathcal{H}_\pi^{-\infty}$  its dual. We denote by  $\mathbb{B}(\mathcal{H}_\pi)_\infty$  the space of smooth operators corresponding to  $\pi$ . This is nothing else than the set smooth vectors for the irreducible representation  $\pi \otimes \bar{\pi}: G \times G \mapsto \mathbb{B}(\mathfrak{S}_2(\mathcal{H}_\pi))$ ,

$$(\pi \otimes \bar{\pi})(g_1, g_2)A = \pi(g_1)A\pi(g_2^{-1}), \quad g_1, g_2 \in G, \quad A \in \mathfrak{S}_2(\mathcal{H}_\pi),$$

with the topology of  $(\mathfrak{S}_2(\mathcal{H}_\pi))^\infty$ . Here  $\mathfrak{S}_2(\mathcal{H}_\pi)$  is the space of Hilbert-Schmidt operators on  $\mathcal{H}_\pi$ .

For  $A \in \mathbb{B}(\mathcal{H}_\pi)_\infty$  we define the coefficients

$$c_A(x) = c_A^\pi(x) = \text{tr}(\pi(x) \circ A) = \text{tr}(A \circ \pi(x)), \quad x \in G.$$

When  $A = (\cdot | \eta)\xi$ ,  $\xi, \eta \in \mathcal{H}_\pi^\infty$  is a projection we simply put  $c_A^\pi = c_{\xi, \eta}^\pi = c_{\xi, \eta}$ , which means

$$c_{\xi, \eta}^\pi(x) = (\pi(x)\xi | \eta), \quad x \in G.$$

In particular, for  $f \in L^1(G)$  we have that

$$\begin{aligned}
(\check{f} * c_A)(x) &= \int_G f(u^{-1})\text{tr}(\pi(u^{-1}x) \circ A)du \\
&= \text{tr}(\pi(x) \circ (A \circ \int_G f(u^{-1})\pi(u^{-1})du)) \\
&= \text{tr}(\pi(x) \circ (A \circ \pi(f))) \\
&= c_{A \circ \pi(f)}(x),
\end{aligned}$$

for every  $x \in G$ . Similarly

$$c_A * \check{f} = c_{\pi(f) \circ A}.$$

We get thus

$$(2.1) \quad \check{f} * c_{\xi, \eta}^\pi = c_{\xi, (\pi(f)^*)\eta}, \quad c_{\xi, \eta}^\pi * \check{f} = c_{\pi(f)\xi, \eta}.$$

Consider an arbitrary fixed

$$\varphi \in (\ker(\pi))^\perp \cap BC^\infty(G).$$

Then, since  $\ker(\pi)$  is a maximal closed ideal in  $L^1(G)$ , it follows that

$$\ker(\pi) = \{f \in L^1(G) \mid \langle \lambda(x)\rho(y)\varphi, f \rangle = 0, \forall x, y \in G\}.$$

Let  $V_\varphi \subseteq \mathcal{P}(G)$  be the subspace of all polynomial functions  $p \in \mathcal{P}(G)$  such that

$$p(\lambda(x)\rho(y)\varphi) \in L^\infty(G) \text{ for all } x, y \in G.$$

Then  $V_\varphi$  is left and right  $G$ -invariant, and  $\mathcal{V}_p \subseteq V_\varphi$  whenever  $p \in V_\varphi$ .

**Theorem 2.1.** *The space  $V_\varphi$  is contained in  $V_{c_{\eta,\eta'}}$  for every  $\eta, \eta' \in \mathcal{H}_\pi^\infty$ . Specifically, if  $p \in V_\varphi$  and  $\{p = p_k, p_{k-1}, \dots, p_1\}$  is a Jordan-Hölder basis for the finite dimensional subspace  $\mathcal{V}_p$ , then there is a constant  $C = C(\varphi)$  and  $q_{\mathcal{H}_\pi^\infty}$  a seminorm on  $\mathcal{H}_\pi^\infty$  such that*

$$\|p_j c_{\eta,\eta'}\|_{L^\infty(G)} \leq C q_{\mathcal{H}_\pi^\infty}(\eta) q_{\mathcal{H}_\pi^\infty}(\eta') \sup\{\|p_i \varphi\|_{L^\infty(G)} \mid j \leq i \leq k\}$$

for every  $\eta, \eta' \in \mathcal{H}_\pi^\infty$  and  $j = 1, \dots, k$ .

*Proof.* Let  $\xi, \xi', \eta, \eta'$  be vectors in  $\mathcal{H}_\pi^\infty$ . There exists a Schwartz function  $F_{\xi,\eta}$  on  $G$  such that  $\pi(\tilde{F}_{\xi,\eta}) = P_{\xi,\eta} = (\cdot \mid \eta)\xi$  and for every  $N \geq 0$  there exist a seminorm  $q_{\mathcal{H}_\pi^\infty}$  on  $\mathcal{H}_\pi^\infty$  and  $C > 0$  such that

$$\sup_G |(1 + |s|)^N \tilde{F}_{\xi,\eta}(s)| \leq q_{\mathcal{H}_\pi^\infty}(\xi) q_{\mathcal{H}_\pi^\infty}(\eta).$$

(See [Ho77], and also [Pe94]). Similarly for other vectors  $\eta, \xi', \eta', \xi$ . Then for any  $f \in L^1(G)$  we have that

$$\pi(\tilde{F}_{\xi,\eta} * f * \tilde{F}_{\eta',\xi'}) = \langle \check{c}_{\eta,\eta'}, f \rangle P_{\xi,\xi'}.$$

Hence, since  $\varphi \in (\ker(\pi))^\perp$ ,

$$\langle F_{\xi,\eta} * \check{\varphi} * F_{\eta',\xi'}, f \rangle = \langle \check{c}_{\eta,\eta'}, f \rangle \langle \check{\varphi}, \tilde{F}_{\xi,\xi'} \rangle \quad \text{for all } f \in L^1(G).$$

Therefore

$$(2.2) \quad \tilde{F}_{\eta',\xi'} * \varphi * \tilde{F}_{\xi,\eta} = \langle \varphi, F_{\xi,\xi'} \rangle c_{\eta,\eta'}.$$

By (1.4) we have

$$\begin{aligned} p_j(t)(\tilde{F}_{\eta',\xi'} * \varphi * \tilde{F}_{\xi,\eta})(t) &= \int_G \tilde{F}_{\eta',\xi'}(s)(p_j(\varphi * \tilde{F}_{\xi',\eta'}))(s^{-1}t)ds \\ &\quad + \sum_{i=j+1}^k \int_G (\check{a}_{i,j} \tilde{F}_{\eta',\xi'})(s)(p_i(\varphi * \tilde{F}_{\xi,\eta}))(s^{-1}t)ds, \end{aligned}$$

and similarly,

$$\begin{aligned} (p_i(\varphi * \tilde{F}_{\xi,\eta}))(s^{-1}t) &= \int_G p_i(y)\varphi(y)\tilde{F}_{\xi,\eta}(y^{-1}s^{-1}t)dy \\ &\quad + \sum_{l=i+1}^k \int_G p_l(y)\varphi(y)(b_{l,i}\tilde{F}_{\xi,\eta})(y^{-1}s^{-1}t)dy. \end{aligned}$$

Hence there are  $N \geq 0$  and  $C, C', C'' > 0$  such that

$$\begin{aligned} |p_j(t)(\tilde{F}_{\eta', \xi'} * \varphi * \tilde{F}_{\xi, \eta})(t)| &\leq C \sup_G (1 + |s|)^N |\tilde{F}_{\eta', \xi'}(s)| \sup_{j \leq i \leq k} \|p_j(\varphi * \tilde{F}_{\xi, \eta})\|_{L^\infty(G)} \\ &\leq C' \sup_G (1 + |s|)^N |\tilde{F}_{\eta', \xi'}(s)| \sup_G (1 + |s|)^N |\tilde{F}_{\xi, \eta}(s)| \sup_{j \leq i \leq k} \|p_j \varphi\|_{L^\infty(G)} \\ &\leq C'' q_{\mathcal{H}_\pi^\infty}(\xi) q_{\mathcal{H}_\pi^\infty}(\xi') q_{\mathcal{H}_\pi^\infty}(\eta) q_{\mathcal{H}_\pi^\infty}(\eta') \sup_{j \leq i \leq k} \|p_j \varphi\|_{L^\infty(G)}, \end{aligned}$$

where  $q_{\mathcal{H}_\pi^\infty}$  is the appropriate seminorm on  $\mathcal{H}_\pi^\infty$ . Using (2.2) with  $\xi, \xi' \in \mathcal{H}_\pi^\infty$  fixed, depending on  $\varphi$  such that  $\langle \varphi, F_{\xi, \xi'} \rangle = 1$ , one gets

$$\begin{aligned} \|p_j c_{\eta, \eta'}\|_{L^\infty(G)} &\leq C''' q_{\mathcal{H}_\pi^\infty}(\xi) q_{\mathcal{H}_\pi^\infty}(\xi') q_{\mathcal{H}_\pi^\infty}(\eta) q_{\mathcal{H}_\pi^\infty}(\eta') \sup\{\|p_i \varphi\|_{L^\infty(G)} \mid j \leq i \leq k\} \\ &\leq C(\varphi) q_{\mathcal{H}_\pi^\infty}(\eta) q_{\mathcal{H}_\pi^\infty}(\eta') \sup\{\|p_i \varphi\|_{L^\infty(G)} \mid j \leq i \leq k\}, \end{aligned}$$

which finishes the proof.  $\square$

**Definition 2.2.** Define the space of polynomial functions  $V_\pi$  by

$$V_\pi := \{p \in \mathcal{P}(G) \mid pc_{\xi, \eta} \in L^\infty(G), \forall \xi, \eta \in \mathcal{H}_\pi^\infty\}.$$

Note that  $V_\pi$  is a left and right invariant subspace of  $\mathcal{P}(G)$ , since

$$(\lambda(x)p)c_{\xi, \eta} = \lambda(x)(p\lambda(x^{-1})(c_{\xi, \eta})) = \lambda(x)(pc_{\xi, \pi(x^{-1})\eta})$$

and  $\pi(x^{-1})\eta \in \mathcal{H}_\pi^\infty$ , and similarly for  $\rho(x)p$ .

**Lemma 2.3.** *Let  $p \in \mathcal{P}(G)$  be arbitrary and fixed. Then for every  $N_0 \in \mathbb{N}$  we have*

$$(2.3) \quad p((\ker(\pi) \cap \mathcal{S}(G))^{N_0 d_p}) \subseteq (\ker(\pi) \cap \mathcal{S}(G))^{N_0}.$$

*Proof.* We prove (2.3) by induction on  $d_p = \dim \mathcal{V}_p$ .

If  $d_p = 1$ , then  $p$  is the constant function and (2.3) holds. Assume now that  $d_p > 1$ . Consider a Jordan-Hölder basis  $\{p_{d_p} = p, \dots, p_1\}$  of  $\mathcal{V}_p$ . Let  $f_i, i = 1, \dots, d_p$  be elements of  $(\ker(\pi) \cap \mathcal{S}(G))^{N_0}$ . We have seen (1.4) that

$$p(f_{d_p} * \dots * f_1) = \sum_{i=1}^{d_p} (\tilde{a}_{i, d_p} f_{d_p}) * (p_i(f_{d_p-1} * \dots * f_1)).$$

Here  $a_{d_p, d_p}$  is the constant function 1 and  $d_{p_i} < d_p$  for  $i < d_p$ . Hence by the induction hypothesis we know that  $p(f_{d_p} * \dots * f_1) \in (\ker(\pi) \cap \mathcal{S}(G))^{N_0}$ , which proves (2.3).  $\square$

The same proof gives the next corollary.

**Corollary 2.4.** *Let  $p \in \mathcal{P}(G)$ . Then for every  $N_0 \in \mathbb{N}$  we have*

$$p((\ker(\pi) \cap C_0(G))^{N_0 d_p}) \subseteq (\ker(\pi))^{N_0}.$$

**Corollary 2.5.** *For every  $p \in \mathcal{P}(G)$  we have that*

$$p((\ker(d\pi))^{d_p N_\pi} * \mathcal{S}(G)) \subseteq j(\pi),$$

where  $d_p$  is the  $G$ -degree of  $p$ .

*Proof.* We prove by induction that for  $D_1, \dots, D_N \in \ker(d\pi)$  and  $h \in C_0^\infty(G)$  we have

$$(2.4) \quad D_1 * \dots * D_N * h \in (\ker(\pi) \cap C_0(G))^N.$$

When  $N = 1$  the assertion is clear.

Assume now that we have proved (2.4) for  $N-1$  and all  $N-1$  sets  $D_1, \dots, D_{N-1} \in \ker(d\pi)$ . For every  $k > 0$  there are a finite number of  $f_j \in C_0^k(G)$  and  $v_j \in U(\mathfrak{g}_C)$  such that

$$\delta = \sum f_j * v_j.$$

Then, by choosing  $k$  such that  $D_1 * f_j \in C_0(G)$ , we get

$$D_1 * D_2 * \dots * D_N * h = \sum D_1 * f_j * v_j * D_2 * \dots * D_N * h$$

where the sum is finite. Since  $v_k * D_2 \in \ker(d\pi)$ , we obtain

$$D_1 * D_2 * \dots * D_N * h \in (\ker(\pi) \cap C_0(G))^N,$$

and this proves (2.4).

From Corollary 2.4 and (1.2) we get that for our  $N_\pi \in \mathbb{N}^*$  and for every  $p \in \mathcal{P}(G)$  we have

$$p((\ker(d\pi))^{N_\pi d_p} * C_0^\infty(G)) \subseteq j(\pi)$$

Since  $C_0^\infty(G)$  is dense in  $\mathcal{S}(G)$ , we get that

$$p((\ker(d\pi))^{N_\pi d_p} * \mathcal{S}(G)) \subseteq j(\pi).$$

□

**Theorem 2.6.** *The closed two-sided ideal*

$$K_\pi := \{f \in L^1(G) \mid \langle pc_{\xi, \eta}, f \rangle = 0, \forall p \in V_\pi, \forall \xi, \eta \in \mathcal{H}_\pi^\infty\}$$

*contains the minimal ideal  $j(\pi)$ .*

*Proof.* Let for  $p \in V_\pi, p \neq 0$ ,  $d_p$  be the dimension of  $\mathcal{V}_p$ . We have seen in the proof of Lemma 2.3 that

$$p((\ker(\pi) \cap \mathcal{S}(G))^{d_p}) \subset \ker(\pi).$$

This shows that  $pc_{\xi, \eta}$  is contained in  $((\ker(\pi) \cap \mathcal{S}(G))^{d_p})^\perp$  and therefore also

$$\langle pc_{\xi, \eta}, (\ker(\pi))^{d_p} \rangle = \{0\}$$

for every  $\eta, \xi \in \mathcal{H}_\pi^\infty$ . Hence

$$\langle pc_{\xi, \eta}, j(\pi) \rangle = \{0\}$$

since  $j(\pi) = (\ker(\pi))^d$  for every  $d \geq N_\pi$ . □

**2.2. A dense subspace of  $j(\pi)$ .** Consider

$$\mathfrak{g} = \mathfrak{g}_n \supset \mathfrak{g}_{n-1} \supset \dots \supset \mathfrak{g}_1 \supset \mathfrak{g}_0 = \{0\}$$

a Jordan-Hölder sequence,  $X_k \in \mathfrak{g}_k \setminus \mathfrak{g}_{k-1}$  a corresponding Jordan-Hölder basis, and  $X_1^*, \dots, X_n^*$  its dual basis of  $\mathfrak{g}^*$ . Let  $e$  be the set of jump indices for  $l$  and this basis, that is,

$$e = \{j \mid 1 \leq j \leq n, X_j \notin \mathfrak{g}_{j-1} + \mathfrak{g}(l)\}.$$

Denote  $\mathfrak{g}_e = \text{span}\{X_j \mid j \in e\}$ , and let  $d = \dim \mathfrak{g}_e = \dim \mathcal{O}$ . We then have  $\mathfrak{g}_e + \mathfrak{g}(l) = \mathfrak{g}$ . If  $e' = \{1, \dots, n\} \setminus e$ , then  $\mathfrak{g}_{e'} = \text{span}\{X_j \mid j \in e'\}$  is also a complement of  $\mathfrak{g}_e$ , isomorphic with  $\mathfrak{g}(l)$ . Let  $e = \{j_1 < \dots < j_d\}$  be the set of jump indices, and denote  $e' = \{l_1 < \dots < l_m\}$ ,  $m + d = n$ .

A set of generators for  $\ker(d\pi)$  were given in [Pe84]: The coadjoint orbit  $\mathcal{O}_\ell$  is given by

$$\mathcal{O}_\ell = \left\{ \sum_{j=1}^n R_j(t) X_j^* \mid t \in \mathbb{R}^d \right\},$$

where the polynomials  $R_j$  have the following properties:



- i)  $R_j$  depends only on the variables  $t_1, \dots, t_k$  where  $j_k \leq j < j_{k+1}$ .
- ii)  $R_{j_k}(t) = t_k$ , for all  $k = 1, \dots, d$ .

Then  $\ker(d\pi)$  is generated by  $u_j \in U(\mathfrak{g}_{\mathbb{C}})$ , where

$$(2.5) \quad u_j = X_j - i\omega(R_j(-iX_{j_1}, \dots, -iX_{j_d}))$$

where  $\omega: S(\mathfrak{g}_{\mathbb{C}}) \rightarrow U(\mathfrak{g}_{\mathbb{C}})$  is the symmetrization map. Note that  $u_j = 0$  when  $j \in e$ .

**Definition 2.7.** We set

$$K_{\pi,0} := \{f \in \mathcal{S}(G) \mid pf \in \ker(\pi), \forall p \in \mathcal{P}(G)\}.$$

*Remark 2.8.* It is easy to see that  $K_{\pi,0}$  is a closed twosided ideal of  $\mathcal{S}(G)$  contained in  $\mathcal{S}(G) \cap \ker(\pi)$ .

**Lemma 2.9.** *The subspace  $K_{\pi,0}$  contains the generators of the minimal ideal  $J(\pi)$ .*

*Proof.* We have seen in Definition 1.1 that the minimal ideal  $J(\pi)$  is generated by Schwartz functions  $f$  for which exist  $g \in \mathcal{S}(G)$  such that  $\pi(g) = 0$  and  $g * f = f * g = f$ . Hence, by Lemma 2.3 for  $p \in \mathcal{P}(G)$  we have that

$$pf = p(g^{d_p} * f) \subset \ker(\pi),$$

therefore  $f \in K_{\pi,0}$ . □

**Lemma 2.10.** *The subspace  $K_{\pi,0}$  is contained in  $j(\pi)$ .*

*Proof.* Let  $X_1, \dots, X_n$  be the Jordan-Hölder basis as above, and recall that we have identified  $G$  and  $\mathfrak{g}$  via the exponential mapping.

On  $G \simeq \mathfrak{g}$  we consider the chart

$$(2.6) \quad \theta: \exp \sum_{j=1}^n x_j X_j \mapsto (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Thus we may assume that  $G = \mathbb{R}^n$  with multiplication given by

$$x \cdot y = \theta(\theta^{-1}(x) \cdot \theta^{-1}(y)).$$

We consider polynomials on  $G$  to be written in the chart  $\theta$ , thus we will identify a  $p \in \mathcal{P}(G)$  with  $p \circ \theta^{-1}$  which is a polynomial function on  $\mathbb{R}^n$ .

Recall that for  $j = 1, \dots, n$ , and  $\varphi \in C^\infty(G)$  we have

$$(2.7) \quad \begin{aligned} (X_j * \varphi)(y) &= (d\lambda(X_j)\varphi)(y) = \frac{d}{ds}\varphi((-sx) \cdot y)|_{s=0} \\ &= -\partial_k \varphi(y) + \sum_{j=1}^{k-1} \alpha_{jk}(y_{j+1}, \dots, y_n) \partial_j \varphi(y), \end{aligned}$$

where  $\alpha_{jk}$  are polynomial functions.

Let  $f \in K_{\pi,0}$ . Then  $f \in \ker(\pi) \cap \mathcal{S}(G) = \ker(d\pi) * \mathcal{S}(G)$  ([dC87, Thm.3.5]), hence there are Schwartz functions  $g_1, \dots, g_m$  such that

$$f = u_{l_m} * g_m + \dots + u_{l_1} * g_1.$$

Let  $p$  a polynomial that depends on  $x_{l_m}$  only. Then by (2.5) and (2.7) we have that

$$pf = -p'g_m + u_{l_m} * (pg_m) + \sum_{j < m} u_{l_j} * (pg_j).$$

If we chose  $p(x) = p_1(x) = x_{l_m}$ , since  $pf \in \ker(\pi)$ , we obtain that

$$g_m = \sum_{j \leq m} u_{l_j} * h_j \in \ker(\pi).$$

Thus it follows that

$$f = u_{l_m}^2 * h_m + \sum_{j < m} u_{l_m} * u_{l_j} * h_j + \sum_{j < m} u_{l_m} * g_j.$$

Since  $u_{l_m} * u_{l_j} \in \sum_{k < m} u_{l_k} * U(\mathfrak{g}_{\mathbb{C}})$  when  $j < m$  (see [Pe84, Prop 2.3.1]), we can write

$$(2.8) \quad f = u_{l_m} * u_{l_m} * g_{m,2} + \sum_{j < m} u_{l_j} * g_{j,2},$$

for some  $g_{k,2} \in \mathcal{S}(G)$ .

We prove by induction that for each  $N$  there are  $g_{k,N} \in \mathcal{S}(G)$  such that

$$(2.9) \quad f = u_{l_m}^N * g_{m,N} + \sum_{j < m} u_{l_j} * g_{j,N}.$$

Since  $f$  has been chosen arbitrary and  $pf$  satisfies the same properties as  $f$  for all  $p \in \mathcal{P}(G)$  we will also have also that

$$pf \in u_{l_m}^N * \mathcal{S}(G) + \sum_{j < m} u_{l_j} * \mathcal{S}(G).$$

We show first by induction over  $k \in \mathbb{N}$

$$(2.10) \quad x_{l_m}^k (u_{l_m}^k * g) = (-1)^k k! g + u_{l_m} * h_k$$

for all  $g \in \mathcal{S}(G)$ , where  $h_k \in \mathcal{S}(G)$ . For  $k = 1$  this has been shown above. If  $k > 1$  denote  $g_{k-1} = u_{l_m}^{k-1} * g$ . Then

$$u_{l_m} * (x_{l_m}^k g_{k-1}) = k x_{l_m}^{k-1} g_{k-1} + x_{l_m}^k (u_{l_m}^k * g).$$

Hence by the induction hypothesis,

$$\begin{aligned} x_{l_m}^k (u_{l_m}^k * g) &= u_{l_m} * (x_{l_m}^k g_{k-1}) - k[(-1)^k (k-1)! g + u_{l_m} * h_{k-1}] \\ &= u_{l_m} * h_k + (-1)^k k! g. \end{aligned}$$

This proves (2.10).

Now assume that we have shown that

$$f = u_{l_m}^k * g_{m,k} + \sum_{j < m} u_{l_j} * g_{j,k}$$

for some  $g_{j,k} \in \mathcal{S}(G)$ . Then if  $p_k(x) = x_{l_m}^k$ , we have by using (2.10),

$$p_k f = (-1)^k k! g_{m,k} + \text{term in } \ker(\pi),$$

hence  $g_{m,k} \in \ker(\pi)$ . Thus (2.9) follows.

Denote by  $\mathcal{P}_k$  polynomials in variable  $x_{l_k}$ ,  $1 \leq k \leq m-1$ , and by  $\mathcal{P}_k^D$  the subset of  $\mathcal{P}_k$  consisting of polynomials of degree  $D$ . Iterating the above procedure we get that for every  $N_m, N_{m-1}, \dots, N_1$  we get that

$$\begin{aligned} (2.11) \quad f &\in \mathcal{P}_1^{\widetilde{N}_1} \dots \mathcal{P}_{m-1}^{\widetilde{N}_{m-1}} ((\ker(d\pi))^{N_m} * \mathcal{S}(G)) \\ &+ \mathcal{P}_1^{\widetilde{N}_1} \dots \mathcal{P}_{m-2}^{\widetilde{N}_{m-2}} ((\ker(d\pi))^{N_{m-1}} * \mathcal{S}(G)) + \dots \\ &\dots + \mathcal{P}_1^{\widetilde{N}_1} ((\ker(d\pi))^{N_2} * \mathcal{S}(G)) + (\ker(d\pi))^{N_1} * \mathcal{S}(G) \end{aligned}$$

for all  $f \in K_{\pi,0}$ , where  $\widetilde{N}_k = 1 + 2 + \cdots + (N_k - 1)$ . Indeed, assume that we have proved that for all  $f \in K_{\pi,0}$  and  $1 < k < m$ ,

$$f = \sum_{j \leq k} u_{l_j} * g_j + \widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} ((\ker(d\pi))^{N_m} * \mathcal{S}(G)) + \cdots + (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G)$$

and thus  $pf$  can be written similarly for every  $p \in \mathcal{P}(G)$ . We have on the other hand

$$\begin{aligned} x_{l_k} f &= -g_k + u_{l_k} * (x_{l_k} g_k) + \sum_{j < k} u_{l_j} * (x_{l_k} g_j) \\ &\quad + x_{l_k} \widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} ((\ker(d\pi))^{N_m} * \mathcal{S}(G)) + \cdots + x_{l_k} (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G) \end{aligned}$$

It follows that  $g_k$  is of the form

$$\begin{aligned} g_k &= u_{l_k} * h_k + \sum_{j < k} u_{l_j} * h_j \\ &\quad + x_{l_k} \widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} (\ker(d\pi))^{N_m} * \mathcal{S}(G) + \cdots + x_{l_k} (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G), \end{aligned}$$

with  $h_j \in \mathcal{S}(G)$ . Replacing this in the formula for  $f$  we get that

$$\begin{aligned} f &= u_{l_k}^2 * h + \sum_{j < k} u_{l_k} * u_{l_j} * h_j \\ &\quad + \widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} (\ker(d\pi))^{N_m} * \mathcal{S}(G) + \cdots + (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G) \\ &\quad + u_{l_k} * (x_{l_k} \widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} (\ker(d\pi))^{N_m} * \mathcal{S}(G) + \cdots + x_{l_k} (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G)). \end{aligned}$$

Note that  $u_{l_k} * (pg) = p(u_{l_k} * g)$  when  $p \in \mathcal{P}_j$ ,  $j > k$ . Hence

$$\begin{aligned} f &= u_{l_k}^2 * g_{k,2} + \sum_{j < k} u_{l_j} * g_{j,2} \\ &\quad + \mathcal{P}_k^1 [\widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} (\ker(d\pi))^{N_m} * \mathcal{S}(G) + \cdots + (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G)] \end{aligned}$$

Assume that we have proved that for all  $f \in K_{\pi,0}$  we have

(2.12)

$$\begin{aligned} f &= u_{l_k}^q * g_{k,q} + \sum_{j < k} u_{l_j} * g_{j,q} \\ &\quad + \mathcal{P}_k^q [\widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} (\ker(d\pi))^{N_m} * \mathcal{S}(G) + \cdots + (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G)] \end{aligned}$$

Then  $x_{l_k}^q f$  can be written similarly. Also (2.12) shows that for some  $\tilde{g}_{k,q} \in \mathcal{S}(G)$ ,

$$\begin{aligned} x_{l_k}^q f &= (-1)^q q! g_{k,q} + u_{l_k}^2 * \tilde{g}_{k,q} + \sum_{j < k} u_{l_j} * (x_{l_k} g_{j,q}) \\ &\quad + x_{l_k}^q \mathcal{P}_k^q (\widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} (\ker(d\pi))^{N_m} * \mathcal{S}(G) + \cdots + (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G)). \end{aligned}$$

Hence  $g_{k,q}$  is of the form

$$\begin{aligned} g_{k,q} &= u_{l_k}^2 * h_k + \sum_{j < k} u_{l_j} * h_j \\ &\quad + x_{l_k}^q \mathcal{P}_k^q [\widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} (\ker(d\pi))^{N_m} * \mathcal{S}(G) + \cdots + (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G)]. \end{aligned}$$

Replacing this in (2.12), and using the fact that

$$\begin{aligned} u_{l_k}^q * (x_{l_k}^q \mathcal{P}_k^{\tilde{q}} [\widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} (\ker(d\pi))^{N_m} * \mathcal{S}(G)] + \cdots + (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G)) \\ \subseteq \mathcal{P}_k^{\tilde{q}+q} [\widetilde{\mathcal{P}_{k+1}^{N_{k+1}}} \cdots \widetilde{\mathcal{P}_{m-1}^{N_{m-1}}} (\ker(d\pi))^{N_m} * \mathcal{S}(G)] + \cdots + (\ker(d\pi))^{N_{k+1}} * \mathcal{S}(G) \end{aligned}$$

we get that (2.12) holds for  $q$  replaced by  $q+1$ .

We have thus proved (2.11).

Now we choose

- $N_1$  large enough such that  $(\ker(d\pi))^{N_1} * \mathcal{S}(G) \subseteq j(\pi)$ ;
- $N_2$  large enough such that  $\mathcal{P}_1^{\tilde{N}_1} ((\ker(d\pi))^{N_2} * \mathcal{S}(G)) \subseteq j(\pi)$ ;
- $\dots$
- $N_m$  large enough such that  $\mathcal{P}_1^{\tilde{N}_1} \cdots \mathcal{P}_{m-1}^{\tilde{N}_{m-1}} ((\ker(d\pi))^{N_m} * \mathcal{S}(G)) \subseteq j(\pi)$ .

Hence we eventually get that  $f \in j(\pi)$ .  $\square$

Inspecting the proof of Lemma 2.10 one sees that it gives us in fact an  $N_0 \in \mathbb{N}$  such that the closed subspace of  $\mathcal{S}(G)$  defined by

$$\{f \in \mathcal{S}(G) \mid pf \in \ker(\pi); \forall p \in \mathcal{P}(G), \dim \mathcal{V}_p \leq N_0, p \text{ real}\}$$

is contained in  $j(\pi)$ . This, along with Lemma 2.9, proves the following fundamental theorem.

**Theorem 2.11.** *There is finite dimensional subspace  $\mathcal{P}_0^\pi = \mathcal{P}_0$  of  $\mathcal{P}(G)$ , invariant under the action of  $G$  by left and right translations, such that the closed ideal  $J_{\pi,0}$  in  $\mathcal{S}(G)$ , defined by*

$$J_{\pi,0} = \{f \in \mathcal{S}(G) \mid pf \in \ker(\pi); \forall p \in \mathcal{P}_0\},$$

*is contained in  $j(\pi) \cap \mathcal{S}(G)$  and it is dense in  $j(\pi)$ .*

### 3. IDEALS WITH HULL $\{\pi\}$

**3.1. Maximal projections.** The following lemma goes along the same lines as [LMP13, Lemma 7.6].

**Lemma 3.1.** *Let  $J$  be a closed ideal in  $\mathcal{S}(G)$  such that  $J \subseteq \ker(\pi) \cap \mathcal{S}(G)$  and assume that there is  $d \in \mathbb{N}$  such that*

$$(3.1) \quad (\ker(\pi) \cap \mathcal{S}(G)/J)^d = \{0\}.$$

*Let  $q$  be a projection in  $\mathcal{S}(G)/(\ker(\pi) \cap \mathcal{S}(G))$ . Then there exists  $\mathbf{q} \in \mathcal{S}(G)/J$  such that  $\mathbf{q} * \mathbf{q} = \mathbf{q}$ ,  $\mathbf{q}^* = \mathbf{q}$ , and, if*

$$\mu: \mathcal{S}(G)/J \longrightarrow \mathcal{S}(G)/(\ker(\pi) \cap \mathcal{S}(G)) = (\mathcal{S}(G)/J)/((\ker(\pi) \cap \mathcal{S}(G))/J)$$

*is the canonical projection, then  $\mu(\mathbf{q}) = q$ .*

*Proof.* Let  $q \in \mathcal{S}(G)/(\ker(\pi) \cap \mathcal{S}(G))$  be fixed. Then consider  $g \in \mathcal{S}(G)/J$  such that  $g^* = g$  and  $\mu(g) = q$ . Since  $\ker(\mu) = (\ker(\pi) \cap \mathcal{S}(G))/J$  and (3.1) holds, it follows that

$$(g - g * g)^d = 0 \quad \text{in } \mathcal{S}(G)/J.$$

On the other hand, by Bezout's theorem, it is easy to see that there exists two polynomials  $\Psi, \Phi \in \mathbb{Q}[t]$  such that  $1 = t^d \Psi(t) + (1-t)^d \Phi(t)$ . Hence

$$(3.2) \quad (t^d \Psi(t))^2 = t^d \Psi(t) - (t - t^2)^d \Psi(t) \Phi(t).$$

Denote  $\psi(t) = t^d \Psi(t)$  and  $\mathbf{q} := \psi(g) \in \mathcal{S}(G)/J$ . Then using (3.2) in the commutative subalgebra generated by  $g$  we get that

$$\mathbf{q} * \mathbf{q} = \mathbf{q} \quad \text{in } \mathcal{S}(G)/J.$$

Also,  $\mathbf{q}$  is self-adjoint since  $\psi$  is real and  $g$  self-adjoint.

Now, writing  $\Psi(X) = \sum_{i=0}^n \psi_i X^i$ , we see that

$$1 = 1^d \Psi(1) + 0 = \sum_{i=0}^n \psi_i.$$

Furthermore, since  $\mu(g^k) = \mu(g) = q$  for every  $k \geq 2$ , we get that

$$\mu(\mathbf{q}) = q^{*d} * \Psi(q) = q * \left( \sum_{i=0}^n \psi_i q^d \right) = \left( \sum_{i=0}^n \psi_i \right) q = q.$$

□

*Remark 3.2.* Recall that the ideal  $J_{\pi,0}$  in Corollary 2.11 is closed in  $\mathcal{S}(G)$  and contained in  $j(\pi) \cap \mathcal{S}(G)$ . Moreover, since  $\mathcal{P}_0$  is finite dimensional, Lemma 2.3 shows that there is a  $d > 0$  such that

$$(\ker(\pi) \cap \mathcal{S}(G))^d \subseteq J_{\pi,0}.$$

Hence we can use Lemma 3.1 for  $J_{\pi,0}$ . Thus for every  $\xi \in \mathcal{H}_\pi^\infty$  we can find  $Q_\xi = Q_\xi^* \in \mathcal{S}(G)$  such that

$$(3.3) \quad Q_\xi * Q_\xi = Q_\xi \bmod J_{\pi,0}, \quad \pi(Q_\xi) = P_{\xi,\xi} = (\cdot | \xi)\xi.$$

**Definition 3.3.** Let  $\xi$  be a smooth vector in  $\mathcal{H}_\pi$ . An element  $Q = Q_\xi \in \mathcal{S}(G)$  is called a *maximal projection* (for  $\xi$ ), if  $Q = Q^*$  is a positive element in the  $C^*$  algebra  $C^*(G)$  such that  $\pi(Q) = P_{\xi,\xi}$ , and  $\pi'(Q) \neq 0$  for any  $\pi' \in \widehat{G}$ .

**Proposition 3.4.** Let  $\xi \in \mathcal{H}_\pi^\infty$ . A maximal projection  $Q = Q_\xi$  always exists such that  $Q * Q - Q \in J_{\pi,0} \subset j(\pi)$ .

*Proof.* Choose any  $Q' = (Q')^* \in \mathcal{S}(G)$  as in Remark 3.2, with  $\pi(Q') = P_{\xi,\xi}$ , and take  $Q_1 = Q' * Q'$ . There exists by [Di77, Prop. 4.7.4] an element  $D \in \mathcal{U}(\mathfrak{g})$ , such that  $d\pi(D) = 0$  and  $d\pi'(D) \neq 0$  for any  $\pi' \in \widehat{G}, \pi' \neq \pi$ . Let  $h_t, t > 0$ , be any heat kernel function for some full Laplacian on  $G$ . The operator  $\pi'(h_t)$  has dense range, and therefore, for an  $N$  is large enough, the function  $f := ((D * h_t) * (D * h_t)^*)^{*N_\pi} \in \mathcal{S}(G)$  has the property that  $f \in J_{\pi,0}(\pi)$  and  $\pi'(f) \neq 0$  for  $\pi' \neq \pi$ . Then  $Q := Q_1 + f$  satisfies the conditions in the statement. □

*Remark 3.5.* For the rest of the paper, when  $\xi \in \mathcal{H}_\pi^\infty$  is fixed,  $Q = Q_\xi$  will be a maximal projection satisfying the conditions of Proposition 3.4.

### 3.2. A correspondence between ideals with hull $\pi$ and the ideals of a finite dimensional algebra.

**Definition 3.6.** (1) Let  $\mathcal{I}^\pi$  be the set of all closed two-sided ideals  $I$  in  $L^1(G)$  with hull  $h(I) = \{\pi\}$ .

(2) Let also  $\mathcal{C}_\pi$  be the vector space of all smooth coefficient functions  $c_A, A \in \mathbb{B}(\mathcal{H}_\pi)_\infty$ .

- (3) Let  $p \in \mathcal{P}_0$  and let  $\{p_m, \dots, p_1 = 1\}$  be a Jordan-Hölder basis of the  $\lambda \otimes \rho$  sub-module  $\mathcal{V}_p$  of  $\mathcal{P}_0$  generated by  $p$ . Then for  $j \in \{0, \dots, m\}$ , let  $\mathcal{V}_p^j C_\pi$  be the space of all functions of the form  $\sum_{i=1}^j p_i \varphi_i$ , where the  $\varphi_i$ ,  $i = m, \dots, 1$  are functions in  $\mathcal{C}_\pi$ .

For  $\varphi \in \ker(\pi)^\perp$ ,  $\delta, \eta \in \mathcal{H}_\pi^\infty$  of norm 1, and  $Q_\eta, Q_\delta$  as in Remark 3.5, let us compute  $Q_\delta * (p_j \varphi) * Q_\eta$  using (1.4), (1.3) and (2.2), as follows:

$$\begin{aligned}
 \check{Q}_\delta * (p_j \varphi) * \check{Q}_\eta &= \sum_{i=1}^j (p_i ((a_{i,j} \check{Q}_\delta) * \varphi)) * \check{Q}_\eta \\
 &= (p_j (\check{Q}_\delta * \varphi)) * \check{Q}_\eta + \sum_{i=1}^{j-1} (p_i ((\check{a}_{i,j} \check{Q}_\delta) * \varphi)) * \check{Q}_\eta \\
 (3.4) \quad &= \sum_{i=1}^j p_i (\check{Q}_\delta * \varphi * b_{i,j} \check{Q}_\eta) + \sum_{i=1}^{j-1} \sum_{k=1}^i p_k ((\check{a}_{i,j} \check{Q}_\delta) * \varphi * (b_{k,i} \check{Q}_\eta)) \\
 &= \langle \delta, \eta \rangle \langle \varphi, \check{Q}_\delta * \check{Q}_\eta \rangle p_j c_{\eta, \delta} + \sum_{i=1}^{j-1} p_i \varphi_i
 \end{aligned}$$

for some  $\varphi_{j-1}, \dots, \varphi_1 \in \ker(\pi)^\perp$ . Furthermore, since  $p_j \varphi \in J_{\pi, 0}^\perp$ ,

$$\check{Q}_\delta * p_j \varphi * \check{Q}_\eta = \check{Q}_\delta * \check{Q}_\delta * (p_j \varphi) * \check{Q}_\eta * \check{Q}_\eta.$$

This shows inductively that

$$(3.5) \quad \check{Q}_\delta * p \varphi * \check{Q}_\eta = \check{Q}_\delta * \tilde{p} c_{\eta, \delta} * \check{Q}_\eta,$$

for some  $\tilde{p} = p + p' \in \mathcal{V}_p$  with  $p' \in \mathcal{V}_p^{m-1}$ .

**Theorem 3.7.** *Let  $\delta, \eta \in \mathcal{H}_\pi^\infty$  of norm 1 and  $I \in \mathcal{I}^\pi$ . Then for any  $\psi \in I^\perp$  there exists  $p \in \mathcal{P}_0$  such that*

$$\check{Q}_\delta * \psi * \check{Q}_\eta = \check{Q}_\delta * (p c_{\eta, \delta}) * \check{Q}_\eta$$

on  $\mathcal{S}(G)$ .

*Proof.* Let  $\omega: G \mapsto [1, \infty[$  be a smooth symmetric polynomial weight, such that  $\mathcal{P}_0$  is bounded by  $\omega$ . (Such a weight always exists, see [LM10].) Then the subspace  $\mathcal{P}_0 \ker(\pi)^\perp$  spanned by the functions  $p \varphi$ ,  $p \in \mathcal{P}_0$ ,  $\varphi \in \ker(\pi)^\perp$  is contained in  $L^\infty(G, \omega)$  and

$$j(\pi) \cap L^1(G, \omega) \supset \{f \in L^1(G, \omega) \mid \langle \psi, f \rangle = 0, \psi \in \mathcal{P}_0 \ker(\pi)^\perp\} \supset J_{\pi, 0}.$$

Let now  $\psi \in j(\pi)^\perp$ . Then  $\psi$  is contained  $L^\infty(G, \omega)$  and thus in the weak\*-limit of  $\text{span}\{p \varphi \mid p \in \mathcal{P}_0, \varphi \in \ker(\pi)^\perp\} \subset L^\infty(G, \omega)$ . Hence  $\check{Q}_\delta * \psi * \check{Q}_\eta$  is contained in the weak\* closure of  $\check{Q}_\delta * (\mathcal{P}_0 c_{\eta, \delta}) * \check{Q}_\eta$  and therefore is contained in  $\check{Q}_\delta * (\mathcal{P}_0 c_{\eta, \delta}) * \check{Q}_\eta$ , because this is a finite dimensional space.  $\square$

*Remark 3.8.* Let  $\psi \in j(\pi)^\perp$  and  $\delta, \eta \in \mathcal{H}_\pi^\infty$ . Then there exists  $p \in \mathcal{P}_0$ , such that

$$(3.6) \quad \check{Q}_\delta * \psi * \check{Q}_\eta = p c_{\eta, \delta} + \sum_{i=1}^{m-1} p_i \psi_i$$

where  $\psi_i \in \mathcal{C}_\pi$  and where  $\{p = p_m, \dots, p_1\}$  is a Jordan-Hölder basis of  $\mathcal{V}_p$ . Indeed by Theorem 3.7, we have this  $p \in \mathcal{P}_0$  such that  $\check{Q}_\delta * \psi * \check{Q}_\eta = \check{Q}_\delta * pc_{\eta,\delta} * \check{Q}_\eta$ . Then by (3.4) we have

$$\begin{aligned}
 \check{Q}_\delta * \psi * \check{Q}_\eta &= \check{Q}_\delta * pc_{\eta,\delta} * \check{Q}_\eta \\
 (3.7) \quad &= \sum_{i=1}^j p_i (\check{Q}_\delta * c_{\eta,\delta} * b_{i,j} \check{Q}_\eta) + \sum_{i=1}^{j-1} \sum_{k=1}^i p_k ((\check{a}_{i,j} \check{Q}_\delta) * c_{\eta,\delta} * (b_{k,i} \check{Q}_\eta)) \\
 &= \sum_{i=1}^j p_i c_{\pi(b_{i,j} \check{Q}_\eta) \eta, \pi(Q_\delta) * \delta} + \sum_{i=1}^{j-1} \sum_{k=1}^i p_k c_{\pi(\check{b}_{k,i} Q_\eta) \eta, \pi(a_{i,j} Q_\delta) * \delta}
 \end{aligned}$$

**Corollary 3.9.** *Let  $\xi \in \mathcal{H}_\pi^\infty$  of norm 1, and  $Q = Q_\xi$ . The vector space  $Q * (\ker(\pi)/j(\pi)) * Q$  is finite dimensional.*

*Proof.* We have seen in Theorem 3.7 that

$$\check{Q} * \mathcal{P}_0 c_{\xi,\xi} * \check{Q} \supset \check{Q} * j(\pi)^\perp * \check{Q} = (Q * (\ker(\pi)/j(\pi)) * Q)^*$$

is finite dimensional. Hence the space  $Q * (\ker(\pi)/j(\pi)) * Q$  is itself finite dimensional.  $\square$

**Definition 3.10.** Let  $Q = Q_\xi$  be as in Remark 3.5.

- i) We denote by  $M^Q$  the span of the subset  $\{f * Q * h \mid h, f \in L^1(G)\}$ .
- ii) For every two-sided closed ideal  $I$  in  $\mathcal{I}^\pi$ , let

$$(3.8) \quad I_Q := Q * (I/j(\pi)) * Q.$$

Then  $I_Q$  is an ideal in the finite dimensional algebra  $\ker(\pi)_Q = Q * \ker(\pi)/j(\pi) * Q$ .

**Proposition 3.11.** *The subspace  $M^Q$  is dense in  $L^1(G)$  modulo  $\ker(\pi)$ .*

*Proof.* Indeed, let  $f \in \mathcal{S}(G)$  and  $\varepsilon > 0$ . There exists an  $N \in \mathbb{N}$  large enough, such that if  $h = h^* \geq 0$  is any self-adjoint smooth function of compact support of norm  $\|h\|_1 = 1$  on  $G$ , then there exists a smooth real valued compactly supported function  $\psi$  on  $\mathbb{R}$  vanishing in a neighbourhood of 0, such that

$$\|\psi\{h\} - h^{*N}\|_1 < \varepsilon,$$

where  $\psi\{h\} = (2\pi)^{-1} \int_{\mathbb{R}} \hat{\psi}(t) e^{*ith} dt$ . The function  $\psi\{f\}$  is contained in the minimal ideal  $j(\emptyset)$  and has the property that  $\pi(\psi\{h\})$  is a self-adjoint finite rank operator. We can now choose  $h$  as above such that  $\|\psi\{h\} * f - f\|_1 < \varepsilon$ . (See [Di60].) The operator  $\pi(\psi\{h\} * f)$  is smooth and of finite rank. Hence we can write

$$\pi(\psi\{h\} * f) = \sum_{\text{finite}} \pi(f_i * Q * g_i)$$

for some finite family  $\{f_i, g_i\}$  of Schwartz-functions. Let  $m := \sum_{\text{finite}} f_i * Q * g_i$ . Then  $k := \psi\{h\} * f - m \in \ker(\pi)$  and

$$\|f - m - k\|_1 < \varepsilon.$$

$\square$

**Proposition 3.12.** *Let  $\xi \in \mathcal{H}_\pi^\infty$  of norm 1,  $Q = Q_\xi$ . Then for every  $I \in \mathcal{I}^\pi$  we have*

$$I = \overline{\text{span}\{L^1(G) * Q * I * Q * L^1(G)\}}$$

*Proof.* Let  $\varphi \in \{L^1(G) * Q * I * Q * L^1(G)\}^\perp$ . It is enough to show that  $\langle \varphi, I \rangle = \{0\}$ .

Let  $0 \neq f \in I$  of norm 1. For  $\varepsilon > 0$ , we choose  $h \in L^1(G)$  such that  $\|h * f - f\|_1 < \varepsilon$ . There is  $h_0 \in \ker(\pi)$  and  $m_1 \in M^Q$  such that

$$\|h - h_0 - m_1\|_1 < \varepsilon.$$

Then

$$\|f - m_1 * f - (h_0 * f)\|_1 < 2\varepsilon,$$

Continuing in this way, we find  $h_1 \in \ker(\pi)$  and  $m_2 \in M^Q$  such that

$$\|h_0 * f - m_2 * h_0 * f - h_1 * h_0 * f\|_1 < 3\varepsilon.$$

Repeating the above argument  $N_\pi$ -times we see that  $f$  can be approximated by elements in  $M^Q * f$  modulo  $(\ker(\pi))^{N_\pi}$ , hence modulo  $j(\pi)$ . The same approximations on the right eventually show that  $f \in \overline{M_Q * I * M_Q} + j(\pi)$ . Now the closed ideal  $\overline{M_Q * j(\pi) * M_Q}$  is contained in  $j(\pi)$  and its hull is reduced to  $\{\pi\}$ , since  $Q$  is maximal. Hence  $\overline{M_Q * j(\pi) * M_Q} = j(\pi)$  and thus we get that  $f \in \overline{M_Q * I * M_Q}$ . Since

$$\langle \varphi, M^Q * I * M^Q \rangle = \{0\}$$

it follows that  $\langle \varphi, f \rangle = \{0\}$  for all  $f \in I$ . We obtain thus that  $\varphi$  vanishes on  $I$ .  $\square$

**Theorem 3.13.** *Let  $\xi \in \mathcal{H}_\pi^\infty$  and let  $Q = Q_\xi \in \mathcal{S}(G)$  be a maximal projection like in Remark 3.5. The mapping  $I \mapsto I_Q$  from the space  $\mathcal{I}^\pi$  of the primary twosided ideals  $I$  with hull  $\pi$  into the space  $\mathbb{I}^Q$  of twosided ideals of the algebra  $\ker(\pi)_Q$  is a bijection.*

*Proof.* Let  $I, I' \in \mathcal{I}^\pi$  be such that  $Q * I / j(\pi) * Q = Q * I' / j(\pi) * Q$ . Then by the proposition above  $I = I'$ .

Let  $K$  be a two-sided ideal in  $\ker(\pi)_Q$ . Define the ideal  $I_K$  by

$$I_K := \overline{\text{span}\{L^1(G) * \tilde{K} * L^1(G)\}},$$

where  $\tilde{K} := \{f \in Q * L^1(G) * Q \mid f \bmod j(\pi) \in K\}$ . Then  $I_K$  is a closed twosided ideal in  $L^1(G)$ . Furthermore the hull of this ideal is reduced to  $\{\pi\}$ , since  $Q$  is maximal. Also, since  $K$  is an ideal, we clearly have that  $K = Q * (I_K / j(\pi)) * Q = (I_K)_Q$ .  $\square$

**Theorem 3.14.** *For every ideal  $I \in \mathcal{I}^\pi$ , the subspace  $I \cap \mathcal{S}(G)$  is dense in  $I$ .*

*Proof.* Choose  $Q = Q_\xi \in \mathcal{S}(G)$  as above. Then there is an ideal  $K$  in  $(\ker(\pi))_Q$  such that  $I = I_K$  as in the proof of Theorem 3.13. We also have that  $\ker(\pi) \cap \mathcal{S}(G)$  is dense in  $\ker(\pi)$ , and  $j(\pi) \cap \mathcal{S}(G)$  is dense in  $j(\pi)$  (see [Lu83b]). Then, since  $\ker(\pi)_Q$  is finite dimensional, we have  $\ker(\pi)_Q = (\ker(\pi)) \cap \mathcal{S}(G)_Q$ . Therefore, denoting by  $\tilde{K}$  the subspace of  $Q * L^1(G) * Q$  defined by

$$\tilde{K} := \{f \in Q * L^1(G) * Q \mid f \bmod j(\pi) \in K\}$$

we see that every  $f \in \tilde{K}$  can be approximated by functions contained in  $\tilde{K} \cap \mathcal{S}(G)$ . Hence every element in  $I = \overline{L^1(G) * \tilde{K} * L^1(G)}$  can also be approximated by functions contained in  $\mathcal{S}(G) * (\tilde{K} \cap \mathcal{S}(G)) * \mathcal{S}(G)$ .  $\square$



**Definition 3.15.** Let  $I \in \mathcal{I}^\pi$ ,  $\xi \in \mathcal{H}_\pi^\infty$  and  $Q = Q_\xi$  as before. Let

$$\mathcal{P}_Q^I := \{p \in \mathcal{P}_0 \mid \text{there exists } \psi \in I^\perp \text{ such that } \check{Q} * \psi * \check{Q} = \check{Q} * pc_{\xi,\xi} * \check{Q} \text{ on } \mathcal{S}(G)\}.$$

Let also

$$\mathcal{P}_Q^\pi = \mathcal{P}_Q^{j(\pi)}.$$

*Remark 3.16.* There may be polynomials  $p \in \mathcal{P}_0$ , such that

$$\check{Q} * pc_{\xi,\xi} * \check{Q} = 0$$

as functional on  $\mathcal{S}(G) \cap \ker(\pi)$ . Therefore we always consider only subspaces  $W$  of  $\mathcal{P}_0$  containing the space

$$\mathcal{P}_Q^0 := \{p \in \mathcal{P}_0 \mid \check{Q} * pc_{\xi,\xi} * \check{Q} = 0 \text{ on } \mathcal{S}(G) \cap \ker(\pi)\}.$$

It follows then by (2.2) that for  $p \in \mathcal{P}_Q^0$  we have that

$$(3.9) \quad \check{Q} * pc_{\xi,\xi} * \check{Q} = \langle pc_{\xi,\xi}, Q \rangle_{c_{\xi,\xi}}.$$

**Proposition 3.17.** For every  $I \in \mathcal{I}^\pi$  and  $\xi \in \mathcal{H}_\pi^\infty$ ,  $Q = Q_\xi$ , we have that

$$\mathcal{P}_Q^I = \{p \in \mathcal{P}_0 \mid \check{Q} * pc_{\xi,\xi} * \check{Q} = 0 \text{ on } I \cap \mathcal{S}(G)\}.$$

In particular,  $\mathcal{P}_Q^0 = \mathcal{P}_Q^{\ker(\pi)}$ .

*Proof.* By definition we have that  $\mathcal{P}_Q^I \subset \{p \in \mathcal{P}_0 \mid \check{Q} * pc_{\xi,\xi} * \check{Q} \text{ on } I \cap \mathcal{S}(G)\}$ . Let now  $p \in \mathcal{P}_0$ , such that  $\check{Q} * pc_{\xi,\xi} * \check{Q} \in \mathcal{S}'(G)$  vanishes on  $I \cap \mathcal{S}(G)$ . Since  $\mathcal{S}(G) \cap I$  is dense in  $I$  by Theorem 3.14, it follows that the finite dimensional spaces  $Q * (I/j(\pi)) * Q$  and  $Q * (I \cap \mathcal{S}(G))/j(\pi) \cap \mathcal{S}(G) * Q$  are isomorphic. Therefore there exists  $\psi \in I^\perp$ , such that  $\check{Q} * \psi * \check{Q} = \check{Q} * \check{Q} * pc_{\xi,\xi} * \check{Q} * \check{Q} = \check{Q} * pc_{\xi,\xi} * \check{Q}$  on  $\ker(\pi) \cap \mathcal{S}(G)$ . By Proposition 3.7, there exist  $p_\psi \in \mathcal{P}_0$  such that  $\check{Q} * \psi * \check{Q} = \check{Q} * p_\psi c_{\xi,\xi} * \check{Q}$  on  $\mathcal{S}(G)$ . Hence

$$p - p_\psi \in \mathcal{P}_Q^0,$$

therefore (by 3.9)  $\check{Q} * (\psi - p_\psi) c_{\xi,\xi} * \check{Q} = \langle (p - p_\psi) c_{\xi,\xi}, Q \rangle_{c_{\xi,\xi}}$ , and thus

$$\check{Q} * pc_{\xi,\xi} * \check{Q} = \check{Q} * (\psi + \langle pc_{\xi,\xi}, Q \rangle_{c_{\xi,\xi}}) * \check{Q}.$$

It follows that  $p \in \mathcal{P}_Q^I$ . □

**Definition 3.18.** For  $x, y \in G$  the function  $\lambda(x)\rho(y)(\check{Q} * pc_{\xi,\xi} * \check{Q})$  is contained in  $I^\perp$  whenever  $p \in \mathcal{P}_Q^I$ . Thus there exists  ${}_x p_y \in \mathcal{P}_Q^I$ , unique modulo  $\mathcal{P}_Q^0$ , such that

$$\check{Q} * ({}_x p_y) c_{\xi,\xi} * \check{Q} = \check{Q} * (\lambda(x)\rho(y)(\check{Q} * pc_{\xi,\xi} * \check{Q})) * \check{Q}$$

on  $\ker(\pi) \cap \mathcal{S}(G)$ . We say that a subspace  $W$  of  $\mathcal{P}_Q^\pi$  is *invariant* if for every  $p \in W$  the polynomials  ${}_x p_y$  are also in  $W$  for every  $x, y \in G$ .

**Theorem 3.19.** Let  $\xi \in \mathcal{H}_\pi^\infty$  and choose a maximal projection  $Q = Q_\xi$  as before. Then there exists an order reversing bijection between the space  $\mathcal{I}^\pi$  and the space  $\mathbb{I}_Q^\pi$  of invariant subspaces of  $\mathcal{P}_Q^\pi$  containing  $\mathcal{P}_Q^0$ .

*Proof.* It suffices to use Theorem 3.13: The orthogonal of every ideal  $K$  contained in  $\ker(\pi)_Q$  is an invariant subspace and every invariant subspace in  $\ker(\pi)_Q$  defines an ideal in  $\ker(\pi)_Q/j(\pi)_Q$ . □

**Definition 3.20.** Consider the space of polynomials

$$W_\pi := V_\pi \cap \mathcal{P}_0.$$

Then  $W_\pi$  is a translation invariant subspace and for every smooth coefficient  $\varphi = c_{\eta,\delta}$  we have that

$$W_\pi \varphi \subset j(\pi)^\perp.$$

**Definition 3.21.** For any translation invariant subspace  $W \subset W_\pi$  let

$$I^W := \{f \in L^1(G) \mid \langle W(\ker(\pi)^\perp), f \rangle = 0\}.$$

Then  $j(\pi) \subset I^W$  is a closed twosided ideal of  $L^1(G)$  contained in  $\mathcal{I}^\pi$ .

For  $I \in \mathcal{I}^\pi$  consider

$$W_I := \{p \in W_\pi \mid pc_{\delta,\eta} \in I^\perp \text{ for all } \delta, \eta \in \mathcal{H}_\pi^\infty\}.$$

Then  $W_I$  is a translation invariant subspace of  $W_\pi$ .

**Theorem 3.22.** *Let  $I \in \mathcal{I}^\pi$ . Then there exists a translation invariant subspace  $W \subset W_\pi$  such that  $I = I^W$  if and only for some  $\xi \in \mathcal{H}_\pi^\infty$  and a maximal  $Q = Q_\xi \in \mathcal{S}(G)$  we have that the space of polynomials  $\mathcal{P}_Q^I$  is translation invariant modulo  $\mathcal{P}_Q^0$ , that is, there exists a translation-invariant subspace  $W_Q \subset W_\pi$  such that  $\mathcal{P}_Q^I = W_Q + \mathcal{P}_Q^0$ .*

*Proof.* Let  $W \subset W_\pi$  be a translation invariant subspace and take  $\xi \in \mathcal{H}_\pi^\infty$ ,  $Q = Q_\xi$  as before. Let  $p \in W$ . Then  $pc_{\xi,\xi} \in (I^W)^\perp$  and therefore also

$$\check{Q} * pc_{\xi,\xi} * \check{Q} \in (I^W)^\perp.$$

There exists then  $p_1 \in \mathcal{P}_Q^{I^W}$  such that

$$\check{Q} * pc_{\xi,\xi} * \check{Q} = \check{Q} * p_1 c_{\xi,\xi} * \check{Q}.$$

This means that  $p - p_1 \in \mathcal{P}_Q^0$ , i.e.,  $p \in \mathcal{P}_Q^{I^W}$ . Therefore  $W + \mathcal{P}_Q^0 \subset \mathcal{P}_Q^{I^W}$ .

For every  $\psi \in (I^W)^\perp$ , the function  $\check{Q} * \psi * \check{Q}$  is contained in  $\check{Q} * Wc_{\xi,\xi} * \check{Q}$ , since  $(I^W)^\perp$  is the weak\*-limit of the span of the functions  $p'c_{\eta,\xi}$ , with  $p' \in W$ ,  $\eta, \delta \in \mathcal{H}_\pi^\infty$ , and since  $\check{Q} * p'c_{\eta,\delta} * \check{Q} \in \check{Q} * Wc_{\xi,\xi} * \check{Q}$ , due to the fact that  $W$  is translation invariant.

Let now  $p \in \mathcal{P}_Q^{I^W}$ . Then there exists  $\psi \in (I^W)^\perp$  such that  $\check{Q} * \psi * \check{Q} = \check{Q} * pc_{\xi,\xi} * \check{Q} \in \check{Q} * Wc_{\xi,\xi} * \check{Q}$ . Hence our  $p \in \mathcal{P}_Q^{I^W}$  is contained in  $W + \mathcal{P}_Q^0$ . Therefore  $\mathcal{P}_Q^{I^W} = W$  modulo  $\mathcal{P}_Q^0$  is translation invariant modulo  $\mathcal{P}_Q^0$ .

Assume now that  $\mathcal{P}_Q^I$  is translation invariant modulo  $\mathcal{P}_Q^0$ . For any  $\psi \in I^\perp$ , there is  $p \in \mathcal{P}_Q^I$  such that

$$(3.10) \quad \check{Q} * pc_{\xi,\xi} * \check{Q} = \check{Q} * \psi * \check{Q}.$$

Since  $\mathcal{P}_Q^I = W_Q + \mathcal{P}_Q^0$ , we can take  $p \in W_Q$  in (3.10). It follows that  $\psi = 0$  on  $Q * I^{W_Q} * Q$ . Since  $\alpha * \psi * \beta \in I^\perp$  for every  $\alpha, \beta \in L^1(G)$  we actually get that  $\psi = 0$  on  $L^1(G) * Q * I^{W_Q} * Q * L^1(G)$ , hence on  $I^{W_Q}$ . Hence  $I^\perp \subseteq (I^{W_Q})^\perp$ .

On the other hand, if  $p \in W_Q$ , then all its derivatives belong also to  $\mathcal{P}_Q^I$  modulo  $\mathcal{P}_Q^0$ . In particular if  $\{p = p_m, \dots, p_1 = 1\}$  is a Jordan-Hölder basis for the  $G \times G$ -module  $W_Q \subset \mathcal{P}_Q^I$  then all the  $p_i$ 's are contained in  $\mathcal{P}_Q^I$  modulo  $\mathcal{P}_Q^0$ . Hence for all  $\alpha, \beta \in \mathcal{S}(G)$  there is  $\psi_{\alpha,\beta} \in I^\perp$  such that

$$\check{Q} * \alpha * pc_{\xi,\xi} * \beta * \check{Q} = \check{Q} * \psi_{\alpha,\beta} * \check{Q}.$$

Therefore  $pc_{\xi,\xi} = 0$  on  $\mathcal{S}(G) * Q * I^\perp * Q * \mathcal{S}(G)$ , and thus  $pc_{\xi,\xi} \in I^\perp$ . Similarly we have that  $p_j c_{\xi,\xi} \in I^\perp$  for every  $j = 1, \dots, m-1$ . It follows that  $p\lambda(x)\rho(y)c_{\xi,\xi} \in I^\perp$  for every  $x, y \in G$ , and since  $p \in W_Q \subseteq W_\pi$ , we get that  $pc_{\eta,\delta} \in I^\perp$  for all  $\eta, \delta \in \mathcal{H}_\pi^\infty$ . Thus we obtain that  $(I^{W_Q})^\perp \subseteq I^\perp$ . This finishes the proof of the theorem.  $\square$

*Remark 3.23.* We can choose the space  $\mathcal{P}_0 \subseteq \mathcal{P}(G)$  such that  $I^{W_\pi} = K_\pi$ .

Indeed, recall that  $W_\pi = V_\pi \cap \mathcal{P}_0$ ,  $K_\pi := \{f \in L^1(G) \mid \langle pc_{\xi,\eta}, f \rangle = 0, \forall p \in V_\pi, \forall \xi, \eta \in \mathcal{H}_\pi^\infty\}$  and that  $\mathcal{P}_0$  in Theorem 2.11 was chosen as

$$\{p \in \mathcal{P}(G) \mid \dim \mathcal{V}_p \leq N_0\}$$

for some  $N_0$  large enough. For  $N \in \mathbb{N}$ ,  $N \geq 0$  denote

$$\mathcal{P}_N := \{p \in \mathcal{P}(G) \mid \dim \mathcal{V}_p \leq N + N_0\}.$$

Then in all the above considerations  $\mathcal{P}_0$  can be replaced by  $\mathcal{P}_N$ .

Let now  $W_{\pi,N} = V_\pi \cap \mathcal{P}_N$ . These are  $G \times G$  invariant subspaces of  $V_\pi$ , and they define a sequence of closed bilateral ideals

$$I_N = \{f \in L^1(G) \mid \langle W_{\pi,N} \mathcal{C}_\pi, f \rangle = 0\}.$$

Then we have

$$K_\pi \subseteq \dots \subseteq I_N \subseteq I_{N-1} \subseteq \dots \subseteq I_0 = I^{W_\pi}.$$

This corresponds to a sequence of ideals

$$(K_\pi)_Q \subseteq \dots \subseteq (I_N)_Q \subseteq (I_{N-1})_Q \subseteq \dots \subseteq (I_0)_Q = (I^{W_\pi})_Q$$

in the finite dimensional algebra  $(\ker(\pi))_Q$ . Hence there must be an  $N_1$  such that  $(I_N)_Q = (I_{N_1})_Q$  for  $N \geq N_1$ , and by Theorem 3.13 it follows that  $I_N = I_{N_1}$  for  $N \geq N_1$ . On the other hand we have that  $K_\pi = \bigcap_{N \geq N_0} I_N$ , therefore we have that

$K_\pi = I_{N_1}$ . We can then replace  $\mathcal{P}_0$  by  $\mathcal{P}_{N_1}$ , and we take into account that the space  $I^{W_\pi}$  for this new  $\mathcal{P}_0$  is precisely  $I_{N_1}$ .

#### 4. $L^\infty(G/N(\pi))$ -INVARIANT IDEALS

Let  $\pi: G \mapsto \mathbb{B}(\mathcal{H}_\pi)$  be an irreducible unitary representation of  $G$ , as before, and let  $\mathcal{O}_\pi$  be the corresponding coadjoint orbit.

**Definition 4.1.** a) Let  $\mathfrak{n} = \mathfrak{n}(\pi)$  be the subalgebra generated by  $\{\mathfrak{g}(\ell)\}_{\ell \in \mathcal{O}_\pi}$ . Then  $\mathfrak{n}(\pi)$  is an ideal since it is a  $G$ -invariant subalgebra. Furthermore, we know that

$$\mathcal{O}_\pi = \mathcal{O}_\pi + \mathfrak{n}(\pi)^\perp,$$

i.e., the coadjoint orbit is saturated with respect to  $\mathfrak{n}(\pi)$ .

b) Let

$$N = N(\pi) = \exp \mathfrak{n}(\pi).$$

Then  $N(\pi)$  is a closed connected normal subgroup of  $G$

c) Choose a subspace  $\mathfrak{x}$  in  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{x} \oplus \mathfrak{n}(\pi)$ . Let  $\mathcal{X} := \exp \mathfrak{x}$ . The group  $G$  is then the topological product of  $\mathcal{X}$  and  $N(\pi)$ .

d) For a function  $p: G \rightarrow \mathbb{C}$  let  $p_{\mathfrak{n}}: G \rightarrow \mathbb{C}$  be defined by

$$p_{\mathfrak{n}}(xn) := p(n), \quad x \in \mathcal{X}, \quad n \in N(\pi).$$

Then let

$$\mathcal{P}_{0,\mathfrak{n}} := \{p_{\mathfrak{n}} \mid p \in \mathcal{P}_0\}.$$

**Definition 4.2.** An ideal  $I \in \mathcal{I}^\pi$  is called  $L^\infty(G/N(\pi))$ -invariant if and only if for all  $f \in I$

$$\varphi f \in I \quad \text{for all } \varphi \in L^\infty(G).$$

We denote by  $\mathcal{I}^{\pi, N(\pi)}$  the collection of all  $I \in \mathcal{I}^\pi$  which are  $L^\infty(G/N(\pi))$ -invariant.

Since  $\mathcal{O}_\pi$  is saturated with respect to  $\mathfrak{n}(\pi)$  we know that the ideals  $\ker(\pi)^j$ ,  $j = 1, 2, \dots$ , and  $j(\pi)$  are  $L^\infty(G/N(\pi))$  invariant. Indeed we have by [Ki62] that

$$\ker(\pi) = \ker(\text{ind}_{N(\pi)}^G \pi|_{N(\pi)}).$$

**Lemma 4.3.** Let  $p = p(s)$  be a polynomial that depends on  $\mathcal{X}$  only. Then  $p\Phi \in \ker(\pi)^\perp \cap BC^\infty(G)$  for every  $\Phi \in \ker(\pi)^\perp \cap BC^\infty(G)$ .

*Proof.* It is enough to show that  $pc_{\pi(\varphi)}^\pi \in \ker(\pi)^\perp$  for all  $\varphi \in \mathcal{S}(G)$ .

By [Pu79, Lemma 2.3] it follows that there is a  $G$ -invariant measure  $d\ell'$  on  $\mathcal{O}_\pi|_{\mathfrak{n}}$  such that

$$\int_{\mathcal{O}_\pi} \hat{\varphi}(\xi) d\ell = \int_{\mathcal{O}_\pi|_{\mathfrak{n}}} \widehat{\varphi_{\mathfrak{n}}}(\ell') d\ell'.$$

Thus we have

$$c_{\pi(\varphi)}^\pi(g) = \text{tr}(\pi(g^{-1})\pi(\varphi)) = C_\pi \int_{\mathcal{O}_\pi|_{\mathfrak{n}}} (\lambda(g^{-1})\widehat{\varphi})|_{\mathfrak{n}}(\ell') d\ell',$$

with  $C_\pi$  a constant. On the other hand we have  $(\lambda(g^{-1})\varphi)|_{\mathfrak{n}}(Y) = \varphi(g \exp Y)$  for  $Y \in \mathfrak{n}$ , thus

$$(\lambda(g^{-1})\widehat{\varphi})|_{\mathfrak{n}}(\ell') = \int_{\mathfrak{n}} e^{-i\langle \ell', Y \rangle} \varphi(g \exp Y) dY, \quad \ell' \in \mathcal{O}_\pi|_{\mathfrak{n}}.$$

Hence, if  $p(sn) = p(s)$  for every  $n \in N_\pi$ ,  $s \in \mathcal{X}$ , we have that

$$p(sn)(\lambda(\widehat{sn}\varphi))|_{\mathfrak{n}}(\ell') = (\lambda(\widehat{sn}\psi))|_{\mathfrak{n}}(\ell'),$$

where  $\psi = p\varphi$ . It follows that  $pc_{\pi(\varphi)}^\pi = c_{\pi(\psi)}^\pi$ , and this finishes the proof of the lemma.  $\square$

**Lemma 4.4.** Let  $I \in \mathcal{I}^{\pi, N(\pi)}$  and  $\psi$  in  $I^\perp \cap C(G)$ . Then the function  $\psi_n$  is also contained in  $I^\perp$ .

*Proof.* Indeed, we have for  $f \in I \cap \mathcal{S}(G)$ ,  $\varphi \in L^\infty(G/N(\pi)) \cap C(G)$ , and  $t \in G$ ,

$$0 = \int_G \psi(g)\varphi(g)f(tg)dg = \int_{G/N(\pi)} \varphi(s) \int_{N(\pi)} f(tsn)\psi(sn)dn ds.$$

Then

$$0 = \int_{N(\pi)} f(tn)\psi(n)dn, \quad \text{for all } t \in G.$$

Hence

$$0 = \int_{G/N(\pi)} \int_{N(\pi)} f(tn)\psi(sn)dndt = \int_G f(g)\psi_{\mathfrak{n}}(g)dg.$$

Therefore  $\psi_{\mathfrak{n}} \in I^\perp$ .  $\square$

**Lemma 4.5.** *Let  $W \subset W_\pi$  be a translation invariant subspace. Then the ideal  $I^W$  is  $L^\infty(G/N(\pi))$ -invariant.*

*Proof.* It follows from [Pe94] (see also the proof of the above Lemma 4.3) that for any  $\varphi \in \mathcal{S}(G/N(\pi))$  and any smooth coefficient  $c_A$  of  $\pi$  we have that the function  $\varphi c_A$  is also a smooth coefficient function of  $\pi$ . Let now  $p \in W$  and  $f \in I^W$ . Then

$$\langle pc_A, f \rangle = 0 \quad \text{for all } A \in \mathbb{B}(\mathcal{H}_\pi)_\infty.$$

Hence for any  $\varphi \in \mathcal{S}(G/N(\pi))$

$$\langle pc_A, \varphi f \rangle = \langle p(\varphi c_A), f \rangle = 0 \quad \text{for all } A \in \mathbb{B}(\mathcal{H}_\pi)_\infty.$$

Thus we get that  $\varphi f \in I^W$ . □

**Proposition 4.6.** *Let  $I \in \mathcal{I}^{\pi, N(\pi)}$ . Then*

$$I^\perp \cap \mathcal{P}_{0, \mathfrak{n}} \mathcal{C}_\pi$$

*is weak\* dense in  $I^\perp$ .*

*Proof.* We know from Proposition 3.12 that the span of the subspaces  $\check{Q}_\delta * I^\perp * \check{Q}_\eta$ , with  $\delta, \eta \in \mathcal{H}_\pi^\infty$  is weak\* dense in  $I^\perp$ . Now we can write for  $\psi \in I^\perp$

$$\check{Q}_\delta * \psi * \check{Q}_\eta = pc_{\eta, \delta} + \sum_{j=1}^{m-1} p_j c_{A_j}$$

for some  $p \in \mathcal{P}_0$  and some smooth operators  $A_j$ ,  $j = 1, \dots, m-1$ , where  $\{p = p_m, \dots, p_1\}$  is a Jordan-Hölder basis of  $\mathcal{V}_p$ . On the other hand for  $x \in \mathcal{X}$ ,  $n \in \mathfrak{n}_\pi$ , we have that

$$\begin{aligned} (\check{Q}_\delta * \psi * \check{Q}_\eta)(xn) &= p(xn)c_{\eta, \delta}(xn) + \sum_{j=1}^{m-1} p_j(xn)c_{A_j}(xn) \\ &= (p(n) + \sum_{j=1}^{m-1} \check{a}_{j, m-1}(x)p_j(n))c_{\eta, \delta}(xn) \\ &\quad + \sum_{j=1}^{m-1} (p_j(n) + \sum_{i=1}^{j-1} \check{a}_{i, j}(x)p_i(n))c_{A_j}(xn) \\ &= p_{\mathfrak{n}}(xn)c_{\eta, \delta}(xn) + \sum_{j=1}^{m-1} (p_j)_{\mathfrak{n}}(xn)c_{B_j}(xn) \end{aligned}$$

for some  $B_j \in \mathbb{B}(\mathcal{H}_\pi)_\infty$ , where we have used Lemma 4.3. Hence  $\check{Q}_\delta * \psi * \check{Q}_\eta \in \mathcal{P}_{0, \mathfrak{n}} \mathcal{C}_\pi$ , which finishes the proof. □

## 5. EXAMPLES

*1. Flat coadjoint orbits.* Assume that  $G$  is a connected simply connected Lie group with Lie algebra  $\mathfrak{g}$ , and  $\ell \in \mathfrak{g}^*$  is such that the corresponding orbit is flat, that is,  $\mathcal{O}_\ell = \ell + \mathfrak{g}(\ell)^\perp$ , or, equivalently,  $\mathfrak{g}(\ell)$  is an ideal in  $\mathfrak{g}$ .

It is well-known that  $\mathcal{O}_\ell$  corresponds to a set  $\{[\pi_\ell]\}$  of spectral synthesis in  $\hat{G}$  (see [HL81]). Let us present below another way of seeing this.

Let  $\mathfrak{g}_0$  be a complement of  $\mathfrak{g}(\ell)$ ,

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}(\ell),$$

and identify  $G$  and  $\mathfrak{g}$  via the exponential mapping. With this notation a polynomial  $p \in \mathcal{P}(G)$  can be written as

$$p(x + s) = \sum x^\alpha p_\alpha(s), \quad x \in \mathfrak{g}_0, s \in \mathfrak{g}(\ell).$$

Here  $x^\alpha$  corresponds to the polynomial  $t_1^{\alpha_1} \cdots t_k^{\alpha_k}$ ,  $k = \dim \mathfrak{g}_0$ , in the coordinates of the first kind, and are linearly independent polynomials that generate  $\mathcal{P}(\mathfrak{g}_0)$ . Since  $\mathcal{O}_\ell$  is flat, for  $p \in \mathcal{P}(G)$ ,  $p(x + s) = \sum x^\alpha p_\alpha(s)$  and  $f \in \mathcal{S}(G)$ , we have that

$$pf \in \ker(\pi_\ell) \Leftrightarrow \int_{\mathcal{O}_\ell} \widehat{pf}(\xi) d\xi = 0.$$

(See [Lu86, Thm. 1].) Here  $\hat{f}$  denotes the Fourier transform on  $\mathfrak{g}$ , and  $d\xi$  is the invariant normalized Liouville measure on  $\mathcal{O}_\ell$ . With the identification  $\mathcal{O} = \ell + \mathfrak{g}_0^*$  this is further equivalent with the fact that

$$\sum q_\alpha(x) \mathcal{F}_{\mathfrak{g}(\ell)}(p_\alpha f)(x, \ell) = 0 \quad \text{for all } x \in \mathfrak{g}_0,$$

where  $\mathcal{F}_{\mathfrak{g}(\ell)}$  is the partial Fourier transform in variable  $s \in \mathfrak{g}(\ell)$ . We get thus that, in this case, the space  $K_{\pi_\ell, 0}$  defined as in Lemma 2.10 consists of all  $f \in \mathcal{S}(G)$  such that

$$\mathbf{D}_{\mathfrak{g}(\ell)} \mathcal{F}_{\mathfrak{g}(\ell)}((p_\alpha f)(x, \ell)) = 0, \quad x \in \mathfrak{g}_0,$$

for every  $\mathbf{D}_{\mathfrak{g}(\ell)}$  differential operator on  $\mathfrak{g}(\ell)$ . The closure of this space in  $L^1(G)$  is nothing else than

$$\{f \in L^1(G) \mid \mathcal{F}_{\mathfrak{g}(\ell)}(p_\alpha f)(x, \ell) = 0, \forall x \in \mathfrak{g}_0\} = \ker(\pi).$$

Thus we find again that  $\{\pi_\ell\}$  is of spectral synthesis.

Note that for the corresponding representation  $\pi = \pi_\ell$  we have that

$$|c_{\xi, \eta}(x + s)| = |c_{\xi, \eta}(x)| \quad \text{for all } s \in \mathfrak{g}(\ell), x \in \mathfrak{g}_0,$$

and

$$c_{\xi, \eta} \in \mathcal{S}(\mathfrak{g}_0).$$

It follows that  $V_\pi$  consists of polynomials  $p$  such that  $p(x + s) = p(x)$ , for all  $x \in \mathfrak{g}_0$ ,  $s \in \mathfrak{g}(\ell)$ . On the other hand, by using Lemma 4.3 for  $p = p(x)$  and  $\xi, \eta \in \mathcal{H}_{\pi_\ell}$  there is an  $f_p \in \mathcal{S}(G)$  such that

$$pc_{\xi, \eta} = c_{\pi_\ell(f_p)}.$$

We immediately see therefore that  $V_{\pi_\ell} \mathcal{C}_{\pi_\ell} \subseteq \ker(\pi_\ell)^\perp$ .

*2. Step 3 nilpotent Lie groups.* The complete description of the structure of primary ideals of nilpotent Lie groups of step 3 can be found in [Lu83a], and we refer to this paper for details of the computations below. Here we briefly show how our present results can be used to find the ideals in  $\mathcal{I}^\pi$  that are of the form  $I^W$  with  $W$  translation invariant subspace in  $V_\pi$ .

Let  $G$  be a step 3 nilpotent Lie group, connected and simply connected, and let  $\mathfrak{g}$  be its Lie algebra. We can assume that

$$[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = \mathfrak{z}(\mathfrak{g}) = \mathbb{R}z$$

the centre  $\mathfrak{z}(\mathfrak{g})$  is one dimensional,  $\ell \in \mathfrak{g}^*$  satisfies  $\langle \ell, \mathfrak{z}(\mathfrak{g}) \rangle \neq 0$ , and  $0 \neq z \in \mathfrak{z}$  is chosen such that  $\langle \ell, z \rangle = 1$ . Let also  $y_1, \dots, y_k$  be a basis of  $[\mathfrak{g}, \mathfrak{g}]$ . Then there exist  $x_1, \dots, x_k \in \mathfrak{g}$  such that

$$[x_j, y_l] = \delta_{jl}z, \quad j, l = 1, \dots, k.$$

Consider now the two-step subalgebra

$$\mathfrak{h} = \{h \in \mathfrak{g} \mid [h, [\mathfrak{g}, \mathfrak{g}]] = 0\},$$

which is also an ideal in  $\mathfrak{g}$ . Then the centre of  $\mathfrak{h}$  is the abelian subalgebra  $\mathfrak{g}_0 = \mathfrak{g}(l) + [\mathfrak{g}, \mathfrak{g}]$ . Note that  $\mathfrak{g}_0$  is an ideal in  $\mathfrak{g}$  and contains  $\mathfrak{g}(\ell)$ .

The representation  $\pi$  associated with  $\ell$  can be realized as

$$\pi = \text{ind}_H^G \pi_0$$

where  $\pi_0$  is the representation of  $H$  associated with  $\ell|_{\mathfrak{h}}$ .

If we denote  $\mathcal{X} = \text{span}\{x_1, \dots, x_k\}$  and  $\mathbf{X} = \exp \mathcal{X}$ , then we can write  $G = \mathbf{X} \cdot H$ .

Below we identify the groups with their Lie algebras, via the exponential mapping. Let now  $f \in L^2(\mathbf{X}, \mathcal{H}_{\pi_0})$ , and for  $x \in \mathbf{X}$ ,  $h \in H$  define  $\tilde{f}(xh) = \pi_0(h^{-1})(f(x))$ . Then we have for  $u \in \mathbf{X}$

$$\begin{aligned} \pi(xh)f(u) &= \tilde{f}(h^{-1}x^{-1}u) = \tilde{f}(x^{-1}uu^{-1}xh^{-1}x^{-1}u) \\ &= \pi_0(u^{-1}xhx^{-1}u)(\tilde{f}(x^{-1}u)) \\ &= e^{i\langle \ell, [u^{-1}x, h] + \frac{1}{2}[u^{-1}x, [u^{-1}x, h]] \rangle} \pi_0(h)(\tilde{f}(x^{-1}u)). \end{aligned}$$

Note that  $[u^{-1}x, h] = [-u + x, h]$  for every  $h$  in  $\mathfrak{h}$ , and

$$x^{-1}u = (u - x)\left(-\frac{1}{2}[x, u] - \frac{1}{6}[x, [x, u]] - \frac{1}{3}[u, [u, x]]\right),$$

and  $(-\frac{1}{2}[x, u] - \frac{1}{6}[x, [x, u]] - \frac{1}{3}[u, [u, x]]) \in \mathfrak{g}_0$ .

Take now  $f = \phi \otimes \xi$ ,  $g = \psi \otimes \eta$ , where  $\phi, \psi \in \mathcal{S}(\mathcal{X})$  and  $\xi, \eta \in \mathcal{H}_{\pi_0}^\infty$ . Also write  $h \in \mathfrak{h}$  as  $h = tns$  where  $n \in [\mathfrak{g}, \mathfrak{g}]$ , and  $s \in \mathfrak{g}(l)/[\mathfrak{g}, \mathfrak{g}]$ . Then

$$\begin{aligned} c_{f,g}^\pi(xtns) &= \langle f, \pi_\ell(xtns)g \rangle \\ &= c_{\xi,\eta}^{\pi_0}(t) \int_{\mathcal{X}} e^{-i\chi(t,s,n,x,u)} \phi(u) \overline{\psi(u-x)} du \end{aligned}$$

where

$$\begin{aligned} \chi(t, s, n, x, u) &= \langle \ell, s+n \rangle + \langle \ell, [u-x, t+n] \rangle + \langle \ell, [u-x, [u-x, t+s+n]]/2 \rangle \\ &\quad - \langle \ell, [x, u]/2 + [x, [x, u]]/6 + [u, [u, x]]/3 \rangle. \end{aligned}$$

Here

$$\begin{aligned} [u-x, [u-x, s+t]] &= [u-x, [u-x, s]] \\ &= [u, [u, s]] + [x, [x, s]] - [x, [u, s]] - [u, [x, s]] \\ &= [u, [u, s]] + [x, [x, s]] - 2[u, [x, s]] + [s, [u, x]]. \end{aligned}$$

Since  $s \in \mathfrak{g}(l)$  we get

$$\begin{aligned} c_{f,g}^\pi(xtns) &= e^{-i\langle \ell, s+n+t \rangle} e^{-i\langle \ell, [x, [x, s]] - [x, n] \rangle/2} c_{\xi,\eta}^{\pi_0}(t) \\ (5.1) \quad &\times \int_{\mathcal{X}} e^{-i\langle \ell, [u, n+t] \rangle} e^{i\langle \ell, [u, [x, s]] \rangle} e^{-i\langle \ell, [u, [u, s]]/2 \rangle} \phi(u) \overline{\psi_1(u, x)} du. \end{aligned}$$

where  $\psi_1(u, x) = e^{-i\langle \ell, [x, u]/2 + [x, [x, u]]/6 + [u, [u, x]]/3 \rangle} \psi(u-t)$ .

Recall that  $\langle \ell, [u, n] \rangle = \langle u, n \rangle$ , the bracket in the right-hand side being the euclidian scalar product. We get by (5.1) that

$$(5.2) \quad c_{f,g}^\pi(xtns) = (2\pi)^{-k} e^{-i\langle \ell, s+n+t \rangle} e^{-i\langle \ell, [x, [x, s]] - [x, n] \rangle / 2} c_{\xi, \eta}^{\pi_0}(t) \times \\ \times \int_{\mathcal{X}^*} \hat{G}(x, n - \ell \circ \text{ad}([x, s]) - \ell \circ \text{ad}(t) - \xi) \hat{\Phi}(s, \xi) d\xi,$$

where, for  $N$  large enough (to be chosen later on),

$$(5.3) \quad G(x, u) = \overline{\psi_1(u, x)} (1 + |u|)^{-N} \\ \Phi(s, u) = e^{-i\langle \ell, [u, [u, s]] / 2 \rangle} \phi(u) (1 + |u|)^N$$

and the Fourier transform is considered in the second variable of these functions.

A computation similar to the one in [Lu83a] shows that

$$(5.4) \quad |\hat{\Phi}(s, \xi)| \leq C(\det(A(s)^2 + 1))^{-1/4},$$

where  $A(s)$  is the bilinear form

$$\langle A(s)u, u \rangle = \langle \ell, [u, [u, s]] \rangle.$$

It remains to estimate the  $L^1$  norm of  $\hat{G}(x, \xi)$ . Note that

$$G(x, u) = e^{i\langle \ell, [x, u] / 2 + [x, [x, u]] / 6 + [u, [u, x]] / 3 \rangle} \overline{\psi(u - x)} (1 + |u|)^{-N}$$

Since  $\psi \in \mathcal{S}(\mathbf{X})$  we get that we can chose  $N$  large enough such that there is  $C$  with

$$|\partial_u^\alpha G(x, u)| \leq C_{\alpha, N} (1 + |u|)^{-k-1}.$$

Hence there is  $C > 0$  such that for all  $x \in \mathcal{X}$

$$(5.5) \quad \|\hat{G}(x, \cdot)\|_{L^1(\mathcal{X}^*)} \leq C.$$

Summing up (5.5) and (5.4) in (5.3) we get that

$$\sup_{x \in \mathcal{X}} |c_{f,g}^\pi(xtsn)| \leq C |c_{\xi, \eta}^{\pi_0}(t)| (\det(A(s)^2 + 1))^{-1/4}.$$

Let  $\omega$  be the weight on  $\mathfrak{g}_0$  defined by  $\omega(s) = (\det(A(s)^2 + 1))^{1/4}$ . Then we have obtained that if a polynomial  $p$  is such that  $|p(xtns)| \leq C\omega(s)^{-1}$ , together with its  $G$ -derivatives, then it belongs to  $V_\pi$ . Combining this with Lemma 4.3, we see that the ideals in  $\mathcal{I}^\pi$  of the form  $I^W$  with  $W \in V_\pi$  correspond to linear, translation invariant subspaces of polynomials on  $\mathfrak{g}_0$  that are bounded by  $\omega$ . By the result in [Lu83a] we actually know that these are all the primary ideals in  $\mathcal{I}^\pi$ .

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